1. $\dfrac{dy}{dx}=\dfrac{y}{x} \;\Rightarrow\; \dfrac{dy}{y}=\dfrac{dx}{x} \;\; [y\neq 0] \;\Rightarrow\; \displaystyle\int\frac{dy}{y}=\int\frac{dx}{x} \;\Rightarrow\; \ln|y|=\ln|x|+C \;\Rightarrow$

$|y|=e^{\ln|x|+C}=e^{\ln|x|}e^{C}=e^{C}|x| \;\Rightarrow\; y=Kx$, where $K=\pm e^{C}$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y=0$ by allowing K to be zero.)

4. $y'=y^2\sin x \;\Rightarrow\; \dfrac{dy}{dx}=y^2\sin x \;\Rightarrow\; \dfrac{dy}{y^2}=\sin x\,dx \;\; [y\neq 0] \;\Rightarrow\; \displaystyle\int\frac{dy}{y^2}=\int\sin x\,dx \;\Rightarrow$

$-\dfrac{1}{y}=-\cos x+C \;\Rightarrow\; \dfrac{1}{y}=\cos x-C \;\Rightarrow\; y=\dfrac{1}{\cos x+K}$, where $K=-C$. $y=0$ is also a solution.

7. $\dfrac{dy}{dt}=\dfrac{te^t}{y\sqrt{1+y^2}} \;\Rightarrow\; y\sqrt{1+y^2}\,dy=te^t\,dt \;\Rightarrow\; \int y\sqrt{1+y^2}\,dy=\int te^t\,dt \;\Rightarrow\; \frac{1}{3}(1+y^2)^{3/2}=te^t-e^t+C$

[where the first integral is evaluated by substitution and the second by parts] $\Rightarrow\; 1+y^2=[3(te^t-e^t+C)]^{2/3} \;\Rightarrow$

$y=\pm\sqrt{[3(te^t-e^t+C)]^{2/3}-1}$

10. $\dfrac{dz}{dt}+e^{t+z}=0 \;\Rightarrow\; \dfrac{dz}{dt}=-e^te^z \;\Rightarrow\; \int e^{-z}\,dz=-\int e^t\,dt \;\Rightarrow\; -e^{-z}=-e^t+C \;\Rightarrow\; e^{-z}=e^t-C \;\Rightarrow$

$\dfrac{1}{e^z}=e^t-C \;\Rightarrow\; e^z=\dfrac{1}{e^t-C} \;\Rightarrow\; z=\ln\left(\dfrac{1}{e^t-C}\right) \;\Rightarrow\; z=-\ln\left(e^t-C\right)$

13. $x\cos x=(2y+e^{3y})\,y' \;\Rightarrow\; x\cos x\,dx=(2y+e^{3y})\,dy \;\Rightarrow\; \int(2y+e^{3y})\,dy=\int x\cos x\,dx \;\Rightarrow$

$y^2+\frac{1}{3}e^{3y}=x\sin x+\cos x+C$ [where the second integral is evaluated using integration by parts].

Now $y(0)=0 \;\Rightarrow\; 0+\frac{1}{3}=0+1+C \;\Rightarrow\; C=-\frac{2}{3}$. Thus, a solution is $y^2+\frac{1}{3}e^{3y}=x\sin x+\cos x-\frac{2}{3}$.

We cannot solve explicitly for y.

18. $\dfrac{dL}{dt}=kL^2\ln t \;\Rightarrow\; \dfrac{dL}{L^2}=k\ln t\,dt \;\Rightarrow\; \displaystyle\int\frac{dL}{L^2}=\int k\ln t\,dt \;\Rightarrow\; -\frac{1}{L}=kt\ln t-\int k\,dt$

[by parts with $u=\ln t$, $dv=k\,dt$] $\Rightarrow\; -\dfrac{1}{L}=kt\ln t-kt+C \;\Rightarrow\; L=\dfrac{1}{kt-kt\ln t-C}$.

$L(1)=-1 \;\Rightarrow\; -1=\dfrac{1}{k-k\ln 1-C} \;\Rightarrow\; C-k=1 \;\Rightarrow\; C=k+1$. Thus, $L=\dfrac{1}{kt-kt\ln t-k-1}$.

21. $u=x+y \;\Rightarrow\; \dfrac{d}{dx}(u)=\dfrac{d}{dx}(x+y) \;\Rightarrow\; \dfrac{du}{dx}=1+\dfrac{dy}{dx}$, but $\dfrac{dy}{dx}=x+y=u$, so $\dfrac{du}{dx}=1+u \;\Rightarrow$

$\dfrac{du}{1+u}=dx \;\; [u\neq -1] \;\Rightarrow\; \displaystyle\int\frac{du}{1+u}=\int dx \;\Rightarrow\; \ln|1+u|=x+C \;\Rightarrow\; |1+u|=e^{x+C} \;\Rightarrow$

$1+u=\pm e^Ce^x \;\Rightarrow\; u=\pm e^Ce^x-1 \;\Rightarrow\; x+y=\pm e^Ce^x-1 \;\Rightarrow\; y=Ke^x-x-1$, where $K=\pm e^C\neq 0$.

If $u=-1$, then $-1=x+y \;\Rightarrow\; y=-x-1$, which is just $y=Ke^x-x-1$ with $K=0$. Thus, the general solution is $y=Ke^x-x-1$, where $K\in\mathbb{R}$.

44. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt} = \left(0.05\ \frac{\text{kg}}{\text{L}}\right)\left(5\ \frac{\text{L}}{\text{min}}\right) + \left(0.04\ \frac{\text{kg}}{\text{L}}\right)\left(10\ \frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\ \frac{\text{kg}}{\text{L}}\right)\left(15\ \frac{\text{L}}{\text{min}}\right)$$
$$= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200}\ \frac{\text{kg}}{\text{min}}$$

Hence, $\int \frac{dy}{130 - 3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t + C$. Because $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$, so $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \quad \Rightarrow \quad \ln|130 - 3y| = -\frac{3}{200}t + \ln 130 = \ln(130e^{-3t/200})$, and $|130 - 3y| = 130e^{-3t/200}$. Since y is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that $130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}(1 - e^{-3t/200})$ kg.

(b) After one hour, $y = \frac{130}{3}(1 - e^{-3\cdot 60/200}) = \frac{130}{3}(1 - e^{-0.9}) \approx 25.7$ kg.

Note: As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

48. (a) According to the hint we use the Chain Rule: $m\frac{dv}{dt} = m\frac{dv}{dx}\cdot\frac{dx}{dt} = mv\frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2} \quad \Rightarrow$

$\int v\,dv = \int \frac{-gR^2\,dx}{(x+R)^2} \quad \Rightarrow \quad \frac{v^2}{2} = \frac{gR^2}{x+R} + C$. When $x = 0$, $v = v_0$, so $\frac{v_0^2}{2} = \frac{gR^2}{0+R} + C \quad \Rightarrow$

$C = \frac{1}{2}v_0^2 - gR \quad \Rightarrow \quad \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{gR^2}{x+R} - gR$. Now at the top of its flight, the rocket's velocity will be 0, and its height will be $x = h$. Solving for v_0: $-\frac{1}{2}v_0^2 = \frac{gR^2}{h+R} - gR \quad \Rightarrow \quad \frac{v_0^2}{2} = g\left[-\frac{R^2}{R+h} + \frac{R(R+h)}{R+h}\right] = \frac{gRh}{R+h} \quad \Rightarrow$

$v_0 = \sqrt{\frac{2gRh}{R+h}}$.

(b) $v_e = \lim_{h\to\infty} v_0 = \lim_{h\to\infty}\sqrt{\frac{2gRh}{R+h}} = \lim_{h\to\infty}\sqrt{\frac{2gR}{(R/h)+1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2 \cdot 32\ \text{ft/s}^2 \cdot 3960\ \text{mi} \cdot 5280\ \text{ft/mi}} \approx 36{,}581\ \text{ft/s} \approx 6.93\ \text{mi/s}$