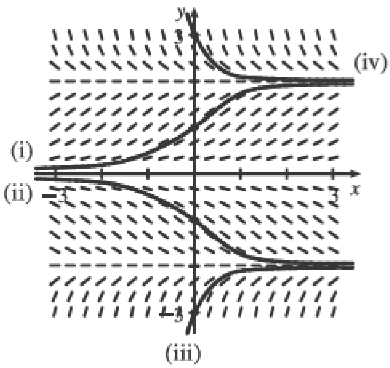
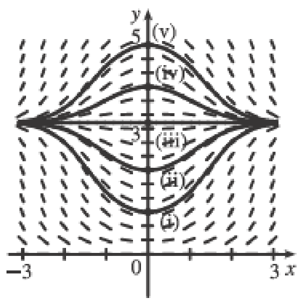
1. (a)




(b) It appears that the constant functions $y = 0$, $y = -2$, and $y = 2$ are equilibrium solutions. Note that these three values of $y$ satisfy the given differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$.

2. (a)




(b) From the figure, it appears that $y = \pi$ is an equilibrium solution. From the equation $y' = x \sin y$, we see that $y = n\pi$ ($n$ an integer) describes all the equilibrium solutions.

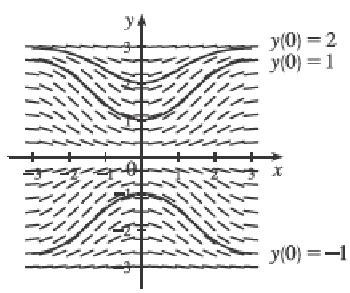
3. $y' = 2 - y$. The slopes at each point are independent of $x$, so the slopes are the same along each line parallel to the $x$-axis. Thus, III is the direction field for this equation. Note that for $y = 2$, $y' = 0$.

4. $y' = x(2 - y) = 0$ on the lines $x = 0$ and $y = 2$. Direction field I satisfies these conditions.

5. $y' = x + y - 1 = 0$ on the line $y = -x + 1$. Direction field IV satisfies this condition. Notice also that on the line $y = -x$ we have $y' = -1$, which is true in IV.

6. $y' = \sin x \sin y = 0$ on the lines $x = 0$ and $y = 0$, and $y' > 0$ for $0 < x < \pi$, $0 < y < \pi$. Direction field II satisfies these conditions.

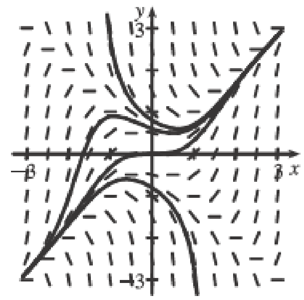
7. (a) $y(0) = 1$
   (b) $y(0) = 2$
   (c) $y(0) = -1$



**10.**

| $x$ | $y$ | $y' = x^2 - y^2$ |
|---|---|---|
| $\pm 1$ | $\pm 3$ | $-8$ |
| $\pm 3$ | $\pm 1$ | $8$ |
| $\pm 1$ | $\pm 0.5$ | $0.75$ |
| $\pm 0.5$ | $\pm 1$ | $-0.75$ |

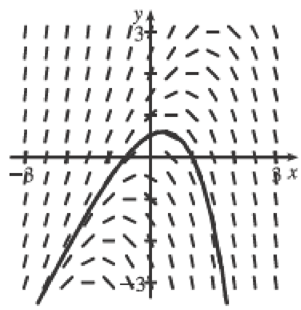
Note that $y' = 0$ for $y = \pm x$. If $|x| < |y|$, then $y' < 0$; that is, the slopes are negative for all points in quadrants I and II above both of the lines $y = x$ and $y = -x$, and all points in quadrants III and IV below both of the lines $y = -x$ and $y = x$. A similar statement holds for positive slopes.




**11.**

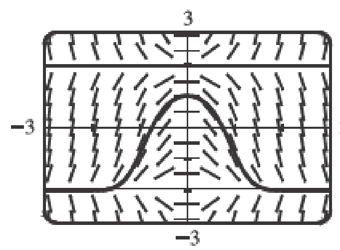
| $x$ | $y$ | $y' = y - 2x$ |
|---|---|---|
| $-2$ | $-2$ | $2$ |
| $-2$ | $2$ | $6$ |
| $2$ | $2$ | $-2$ |
| $2$ | $-2$ | $-6$ |

Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1, 0)$.




**16.** See Exercise 15 for specific CAS directions. The exact solution is

$$y = \frac{2\left(3 - e^{2x^2}\right)}{e^{2x^2} + 3}$$



**19.** (a) $y' = F(x, y) = y$ and $y(0) = 1 \quad \Rightarrow \quad x_0 = 0, y_0 = 1.$

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \quad \Rightarrow \quad y_1 = 1 + 0.4 \cdot 1 = 1.4.$ $x_1 = x_0 + h = 0 + 0.4 = 0.4,$

so $y_1 = y\,(0.4) = 1.4.$

(ii) $h = 0.2 \quad \Rightarrow \quad x_1 = 0.2$ and $x_2 = 0.4$, so we need to find $y_2$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2,$

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44.$

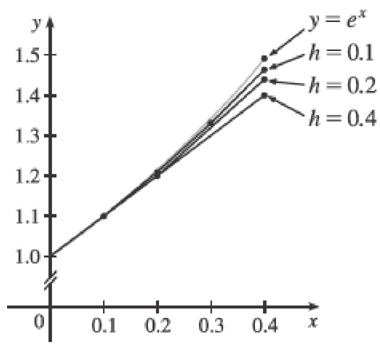
(iii) $h = 0.1 \quad \Rightarrow \quad x_4 = 0.4$, so we need to find $y_4$. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1,$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21,$

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331,$

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641.$

(b)




We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

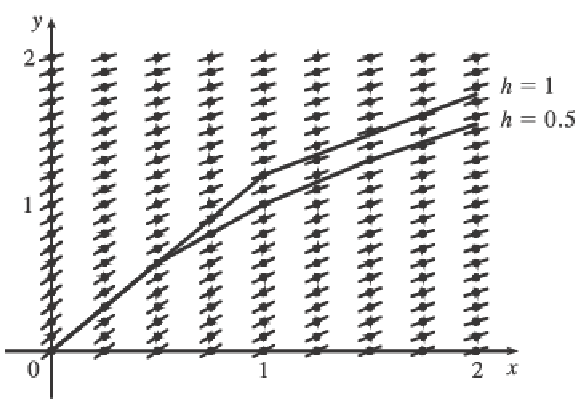
(c) (i) For $h = 0.4$: (exact value) − (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) − (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) − (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

**20.**




As $x$ increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

**25.** (a) $dy/dx + 3x^2y = 6x^2 \quad \Rightarrow \quad y' = 6x^2 - 3x^2y$. Store this expression in $\mathrm{Y}_1$ and use the following simple program to evaluate $y(1)$ for each part, using $\mathrm{H} = h = 1$ and $\mathrm{N} = 1$ for part (i), $\mathrm{H} = 0.1$ and $\mathrm{N} = 10$ for part (ii), and so forth.

$h \to$ H: 0 $\to$ X: 3 $\to$ Y:

For(I, 1, N): Y + H $\times$ $\mathrm{Y}_1$ $\to$ Y: X + H $\to$ X:

End(loop):

Display Y. [To see all iterations, include this statement in the loop.]

(i) $\mathrm{H} = 1, \mathrm{N} = 1 \quad \Rightarrow \quad y(1) = 3$
(ii) $\mathrm{H} = 0.1, \mathrm{N} = 10 \quad \Rightarrow \quad y(1) \approx 2.3928$
(iii) $\mathrm{H} = 0.01, \mathrm{N} = 100 \quad \Rightarrow \quad y(1) \approx 2.3701$
(iv) $\mathrm{H} = 0.001, \mathrm{N} = 1000 \quad \Rightarrow \quad y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \quad \Rightarrow \quad y' = -3x^2e^{-x^3}$

$$\mathrm{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \mathrm{RHS}$$

$y(0) = 2 + e^{-0} = 2 + 1 = 3$

(c) The exact value of $y(1)$ is $2 + e^{-1^3} = 2 + e^{-1}$.
(i) For $h = 1$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 3 \approx -0.6321$
(ii) For $h = 0.1$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3928 \approx -0.0249$
(iii) For $h = 0.01$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3701 \approx -0.0022$
(iv) For $h = 0.001$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3681 \approx -0.0002$

In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).