4. (a) $y = \cos kt \Rightarrow y' = -k \sin kt \Rightarrow y'' = -k^2 \cos kt$. Substituting these expressions into the differential equation $4y'' = -25y$, we get $4(-k^2 \cos kt) = -25(\cos kt) \Rightarrow (25 - 4k^2) \cos kt = 0$ [for all t] $\Rightarrow 25 - 4k^2 = 0 \Rightarrow k^2 = \frac{25}{4} \Rightarrow k = \pm\frac{5}{2}$.

(b) $y = A \sin kt + B \cos kt \Rightarrow y' = Ak \cos kt - Bk \sin kt \Rightarrow y'' = -Ak^2 \sin kt - Bk^2 \cos kt$.

The given differential equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Thus,

$$\begin{aligned} \text{LHS} = 4y'' + 25y &= 4(-Ak^2 \sin kt - Bk^2 \cos kt) + 25(A \sin kt + B \cos kt) \\ &= -4Ak^2 \sin kt - 4Bk^2 \cos kt + 25A \sin kt + 25B \cos kt \\ &= (25 - 4k^2)A \sin kt + (25 - 4k^2)B \cos kt \\ &= 0 \quad \text{since } k^2 = \tfrac{25}{4}. \end{aligned}$$

5. (a) $y = \sin x \Rightarrow y' = \cos x \Rightarrow y'' = -\sin x$.

LHS $= y'' + y = -\sin x + \sin x = 0 \neq \sin x$, so $y = \sin x$ is **not** a solution of the differential equation.

(b) $y = \cos x \Rightarrow y' = -\sin x \Rightarrow y'' = -\cos x$.

LHS $= y'' + y = -\cos x + \cos x = 0 \neq \sin x$, so $y = \cos x$ is **not** a solution of the differential equation.

(c) $y = \frac{1}{2}x \sin x \Rightarrow y' = \frac{1}{2}(x \cos x + \sin x) \Rightarrow y'' = \frac{1}{2}(-x \sin x + \cos x + \cos x)$.

LHS $= y'' + y = \frac{1}{2}(-x \sin x + 2 \cos x) + \frac{1}{2}x \sin x = \cos x \neq \sin x$, so $y = \frac{1}{2}x \sin x$ is **not** a solution of the differential equation.

(d) $y = -\frac{1}{2}x \cos x \Rightarrow y' = -\frac{1}{2}(-x \sin x + \cos x) \Rightarrow y'' = -\frac{1}{2}(-x \cos x - \sin x - \sin x)$.

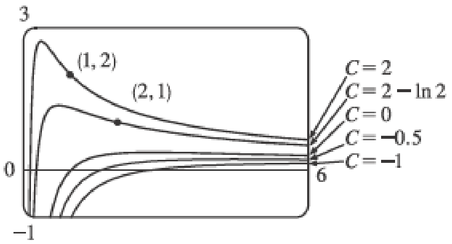
LHS $= y'' + y = -\frac{1}{2}(-x \cos x - 2 \sin x) + \left(-\frac{1}{2}x \cos x\right) = \sin x =$ RHS, so $y = -\frac{1}{2}x \cos x$ **is** a solution of the differential equation.

6. (a) $y = \dfrac{\ln x + C}{x} \quad \Rightarrow \quad y' = \dfrac{x \cdot (1/x) - (\ln x + C)}{x^2} = \dfrac{1 - \ln x - C}{x^2}$.

$$\text{LHS} = x^2 y' + xy = x^2 \cdot \frac{1 - \ln x - C}{x^2} + x \cdot \frac{\ln x + C}{x}$$

$= 1 - \ln x - C + \ln x + C = 1 = \text{RHS}$, so y is a solution of the differential equation.

(b)



A few notes about the graph of $y = (\ln x + C)/x$:

(1) There is a vertical asymptote of $x = 0$.

(2) There is a horizontal asymptote of $y = 0$.

(3) $y = 0 \quad \Rightarrow \quad \ln x + C = 0 \quad \Rightarrow \quad x = e^{-C}$, so there is an x-intercept at e^{-C}.

(4) $y' = 0 \quad \Rightarrow \quad \ln x = 1 - C \quad \Rightarrow \quad x = e^{1-C}$, so there is a local maximum at $x = e^{1-C}$.

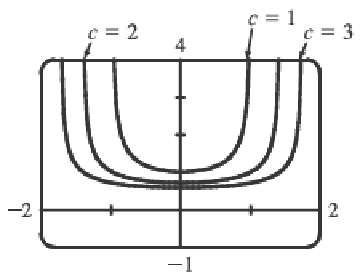
(c) $y(1) = 2 \quad \Rightarrow \quad 2 = \dfrac{\ln 1 + C}{1} \quad \Rightarrow \quad 2 = C$, so the solution is $y = \dfrac{\ln x + 2}{x}$ [shown in part (b)].

(d) $y(2) = 1 \quad \Rightarrow \quad 1 = \dfrac{\ln 2 + C}{2} \quad \Rightarrow \quad 2 + \ln 2 + C \quad \Rightarrow \quad C = 2 - \ln 2$, so the solution is $y = \dfrac{\ln x + 2 - \ln 2}{x}$ [shown in part (b)].

8. (a) If x is close to 0, then xy^3 is close to 0, and hence, y' is close to 0. Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y.)

(b) $y = (c - x^2)^{-1/2} \quad \Rightarrow \quad y' = x(c - x^2)^{-3/2}$. $\text{RHS} = xy^3 = x[(c - x^2)^{-1/2}]^3 = x(c - x^2)^{-3/2} = y' = \text{LHS}$

(c)



When x is close to 0, y' is also close to 0.

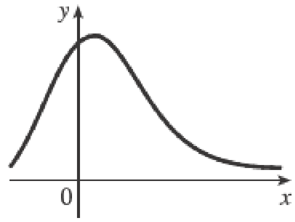
As x gets larger, so does $|y'|$.

(d) $y(0) = (c - 0)^{-1/2} = 1/\sqrt{c}$ and $y(0) = 2 \quad \Rightarrow \quad \sqrt{c} = \frac{1}{2} \quad \Rightarrow \quad c = \frac{1}{4}$, so $y = \left(\frac{1}{4} - x^2\right)^{-1/2}$.

11. (a) This function is increasing *and* also decreasing. But $dy/dt = e^t(y - 1)^2 \geq 0$ for all t, implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When $y = 1$, $dy/dt = 0$, but the graph does not have a horizontal tangent line.

12. The graph for this exercise is shown in the figure at the right.



A. $y' = 1 + xy > 1$ for points in the first quadrant, but we can see that $y' < 0$ for some points in the first quadrant.

B. $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$.

Thus, equations A and B are incorrect, so the correct equation is C.

C. $y' = 1 - 2xy$ seems reasonable since:

(1) When $x = 0$, y' could be 1.

(2) When $x < 0$, y' could be greater than 1.

(3) Solving $y' = 1 - 2xy$ for y gives us $y = \dfrac{1 - y'}{2x}$. If y' takes on small negative values, then as $x \to \infty$, $y \to 0^+$, as shown in the figure.

14. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) $\dfrac{dy}{dt} = k(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y(0) = 95°C$. The answer and the model support each other because as y approaches R, dy/dt approaches 0, so the model seems appropriate.

(c)