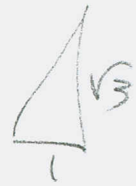
202 §8.5 II #s 67, 72, 77

(67) $\int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x^2}\,dx$  Let $\tan\theta = x$, $dx = \sec^2\theta\, d\theta$

$x = 1 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$

$x = \sqrt{3} = \tan\theta \Rightarrow \theta = \frac{\pi}{3}$




$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}\sec^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\ \tan^{-2}\theta\, \sec^2\theta\, d\theta$$

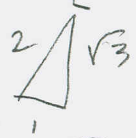
$u = \sec\theta$  $du = \sec\theta\tan\theta\, d\theta$

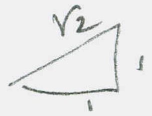
$dv = \tan^{-2}\theta\sec^2\theta\, d\theta$

$v = -\tan^{-1}\theta = -\cot\theta$

$$= uv - \int v\,du = -\Big[\sec\theta\cot\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\cot\theta\sec\theta\tan\theta\, d\theta$$

$$= -\Big[\sec\tfrac{\pi}{3}\cot\tfrac{\pi}{3} - \sec\tfrac{\pi}{4}\cot\tfrac{\pi}{4}\Big] + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\, d\theta$$




$$= -\Big[2\cdot\frac{1}{\sqrt{3}} - \sqrt{2}\cdot 1\Big] + \Big[\ln|\sec\theta + \tan\theta|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\frac{2}{\sqrt{3}} + \sqrt{2} + \Big[\ln\left|\sec\tfrac{\pi}{3} + \tan\tfrac{\pi}{3}\right| - \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right|\Big]$$

$$= \boxed{\frac{-2+\sqrt{6}}{\sqrt{3}} + \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1)}$$

202 §8.5 IV #s 72, 77

72 $\displaystyle\int \frac{4^x+10^x}{2^x}\,dx = \int\left(\frac{4^x}{2^x}+\frac{10^x}{2^x}\right)dx$

$= \displaystyle\int\left(\left(\frac{4}{2}\right)^x+\left(\frac{10}{2}\right)^x\right)dx = \int\left(2^x+5^x\right)dx$

$= \boxed{\dfrac{1}{\ln 2}2^x + \dfrac{1}{\ln 5}5^x + C}$

77 $\displaystyle\int \frac{\sqrt{x}\,dx}{1+x^3}$ Let $u = x^3+1$ Then $x = (u-1)^{\frac{1}{3}}$

and $dx = \frac{1}{3}(u-1)^{-\frac{2}{3}}du$

and $\sqrt{x} = \left((u-1)^{\frac{1}{3}}\right)^{\frac{1}{2}} = (u-1)^{\frac{1}{6}}$

$= \displaystyle\int \frac{(u-1)^{\frac{1}{6}}}{u}\cdot\frac{1}{3}(u-1)^{-\frac{2}{3}}du = \frac{1}{3}\int\frac{(u-1)^{\frac{1}{6}-\frac{2}{3}}du}{u}$

$= \displaystyle\frac{1}{3}\int\frac{(u-1)^{-\frac{5}{6}}du}{u}$ Ouch!

Let $u=\sqrt{x}$. Then $u^2 = x$, which gives $dx = 2u\,du$ and

so $\displaystyle\int\frac{u\cdot 2u\,du}{1+u^6} = \int\frac{2u^2du}{1+u^6}$

Let $v = u^{\frac{3}{2}}$. Then $u = v^{\frac{2}{3}}$ and $u^2 = v^{\frac{4}{3}}$ and $u^6 = v^4$

$du = \frac{2}{3}v^{-\frac{1}{3}}dv$

This gives $\displaystyle 2\int\frac{v^{\frac{4}{3}}}{1+v^4}\cdot\frac{2}{3v^{\frac{1}{3}}}dv$

$= \displaystyle\frac{4}{3}\int\frac{v}{1+v^4}dv.$ Hmmmm

202 §8.5 II #77

$\frac{4}{3}\int \frac{v}{1+v^4}\,dv$ Try Parts: $u=v$ $du=dv$
ucky letters

$= \frac{4}{3}\int \frac{s}{1+s^4}\,ds$ Let $u=s$ $du=ds$
$dv = \frac{ds}{1+s^4}$ Let $r=s^2+1$
$dr = 2s\,ds$ and
$ds = \frac{dr}{2s} = \frac{dr}{2\sqrt{r-1}}$

No Fun.

Let $u=s^2$. Then $du = 2s\,ds$
and so $ds = \frac{du}{2s}$

Yes!

This gives

$$\frac{4}{3}\int \frac{s}{1+u^2}\cdot\frac{du}{2s} = \frac{2}{3}\int \frac{du}{1+u^2} = \frac{2}{3}\arctan(u)+C$$

Now, back-track.

$$= \frac{2}{3}\arctan(s^2)+C = \frac{2}{3}\arctan(v^2)+C$$

$$= \frac{2}{3}\arctan\left(\left(u^{\frac{3}{2}}\right)^2\right)+C = \frac{2}{3}\arctan(u^3)+C$$

$$= \frac{2}{3}\arctan\left(\left(x^{\frac{1}{2}}\right)^3\right)+c = \boxed{\frac{2}{3}\arctan\left(x^{\frac{3}{2}}\right)+C}$$

Book just did the genius "Let $u = x^{\frac{3}{2}}$ and it came out in one step

202 $\S 8.5$ $\underline{\text{IV}}$ #s 42, 47, 52, 57, 62, 67, 72, 77

42 $\displaystyle\int \frac{\tan^{-1}(x)}{x^2}\,dx$ $\qquad u = \arctan(x) \Rightarrow du = \dfrac{dx}{1+x^2}$

$\qquad dv = x^{-2}dx \Rightarrow v = -x^{-1}$

$$= uv - \int v\,du = -\frac{1}{x}\arctan(x) - \int -\frac{1}{x}\cdot\frac{dx}{1+x^2}$$

$$= -\frac{1}{x}\arctan(x) + \int \frac{dx}{x(x^2+1)}$$ Partial Fractions

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

$1 = A(x^2+1) + Bx^2 + Cx$ This gives

$1 = Ax^2 + A + Bx^2 + Cx$

$\boxed{1 = A}$

$$\int \frac{dx}{x(x^2+1)} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$$

$0 = A + B$

$\boxed{B = -1}$

$$= \ln|x| - \arctan(x) + C.$$

$\boxed{0 = C}$

Put it together with the 1st piece, for our final answer:

$$\int \frac{\arctan(x)}{x^2}\,dx = \boxed{-\frac{1}{x}\arctan(x) + \ln|x| - \arctan(x) + C}$$

202 §8.5 IV #s 47, 52, 57, 62, 67, 72, 77

47 $\displaystyle\int x^3(x-1)^{-4}\,dx$ Let $u = x-1$. Then $x = u+1$

$$= \int (u+1)^3 (u)^{-4}\,du$$

$$= \int \frac{u^3+3u^2+3u+1}{u^4}\,du = \int \left(u^{-1}+3u^{-2}+3u^{-3}+u^{-4}\right)du$$

$$= \ln|u| - 3u^{-1} - \frac{3}{2}u^{-2} - \frac{1}{3}u^{-3} + C$$

$$= \boxed{\ln|x-1| - \frac{3}{x-1} - \frac{3}{2}\cdot\frac{1}{(x-1)^2} - \frac{1}{3}\cdot\frac{1}{(x-1)^3} + C}$$

52 $\displaystyle\int \frac{dx}{x(x^4+1)}$ Let $u = x^2$. Then $du = 2x\,dx$

$\sqrt{u} = x$

$$dx = \frac{1}{2x}\,du = \frac{1}{2\sqrt{u}}\,du$$

$$= \int \frac{\frac{1}{2\sqrt{u}}}{\sqrt{u}\,(u^2+1)}\,du$$

$$= \frac{1}{2}\int \frac{du}{u(u^2+1)}$$

$$\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}$$

$$1 = A(u^2+1) + (Bu+C)(u)$$

$$1 = Au^2 + A + Bu^2 + Cu$$

$$\boxed{1 = A}$$

$$0 = A+B \rightarrow \boxed{B=-1}$$

$$0 = C$$

This gives

$$\frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\cdot\frac{1}{2}\int \frac{2u\,du}{u^2+1}$$

$$= \frac{1}{2}\left[\ln|u| - \frac{1}{2}\ln|u^2+1|\right] + C$$

$$= \boxed{\frac{1}{2}\left[\ln|x^2| - \frac{1}{2}\ln|x^4+1|\right] + C}$$

=

202 §8.5 II #s 57, 62, 67, 72, 77

(57) $\int x \sqrt[3]{x+c}\, dx$ Finally got to the one Beth asked about.

Beth did an integration by parts in which I could find no errors, nor could I resolve her work with that of the textbook method.

Beth did this:

$u = x \implies du = dx$

$dv = (x+c)^{\frac{1}{3}}dx \implies v = \frac{3}{4}(x+c)^{\frac{4}{3}}$

So, $uv - \int v\, du = \frac{3}{4}x(x+c)^{\frac{4}{3}} - \frac{3}{4}\int (x+c)^{\frac{4}{3}}dx$

$= \frac{3}{4}x(x+c)^{\frac{4}{3}} - \frac{3}{4}\cdot\frac{3}{7}(x+c)^{\frac{7}{3}} + K$ Looks GOOD 2 me.

$= \frac{3}{4}(x+c)^{\frac{4}{3}}\left[x - \frac{3}{7}(x+c)\right] + K = \frac{3}{4}(x+c)^{\frac{4}{3}}\left[\frac{4}{7}x - \frac{3}{7}c\right] + K$

$= \frac{3}{7}x(x+c)^{\frac{4}{3}} - \frac{9}{28}c(x+c)^{\frac{4}{3}} + K$

Hmmmmmmm

Book Way: Let $u = (x+c)^{\frac{1}{3}}$. Then $du = \frac{1}{3}(x+c)^{-\frac{2}{3}}dx$

and $u^3 = x+c \implies u^3 - c = x$

and so $dx = 3u^2 du$

This gives $\int (u^3 - c)u \cdot 3u^2 du = 3\int (u^6 - cu^3)\, du$

$= 3\frac{u^7}{7} - 3c\frac{u^4}{4} + C = \boxed{\frac{3}{7}(x+c)^{\frac{7}{3}} - \frac{3}{4}c(x+c)^{\frac{4}{3}} + C}$

202 §8.5 II #s 57, 62, 67, 72, 77

57 cont'd Let's see if we can manipulate Beth's result:

$$\frac{3}{7}x(x+c)^{\frac{4}{3}} - \frac{9}{28}c(x+c)^{\frac{4}{3}} + K$$

$$= \frac{3}{7}(x+c-c)(x+c)^{\frac{4}{3}} - \frac{9}{28}c(x+c)^{\frac{4}{3}} + K$$

$$= \frac{3}{7}(x+c)(x+c)^{\frac{4}{3}} - \frac{3}{7}c(x+c)^{\frac{4}{3}} - \frac{9}{28}c(x+c)^{\frac{4}{3}} + K$$

$$= \frac{3}{7}(x+c)^{\frac{7}{3}} - \left(\frac{3}{7}+\frac{9}{28}\right)c(x+c)^{\frac{4}{3}} + K$$

$$= \frac{3}{7}(x+c)^{\frac{7}{3}} - \left(\frac{21}{28}\right)c(x+c)^{\frac{4}{3}} + K$$

$$= \frac{3}{7}(x+c)^{\frac{7}{3}} - \frac{3}{4}c(x+c)^{\frac{4}{3}} + K \quad \text{Cool.}$$

62 $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,dx$ Let $u=\sqrt[3]{x} \longrightarrow x=u^3$

$\longrightarrow dx = 3u^2\,du$

$$= \int \frac{3u^2\,du}{u^3+u} = 3\int \frac{u^2}{u(u^2+1)}\,du = 3\int \frac{u}{u^2+1}\,du$$

$$= \frac{3}{2}\int \frac{2u\,du}{u^2+1} = \boxed{\frac{3}{2}\ln(u^2+1) + C}$$