202 §8.4 #s 4, 9, 20, 21, 27, 34, 40, 45, 46, 48

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

#s 1–6 Show the 1st step in partial fractions decomposition.

4 (a) $$\frac{x^3}{x^2+4x+3} = x-4+\frac{13x+12}{(x+3)(x+1)}$$

$(x+3)(x+1)$

OOPS! Went too far!

$$\begin{array}{r}
x-4 \phantom{+0x+0} \\
x^2+4x+3 \,\overline{)\, x^3+0x^2+0x+0} \\
-(x^3+4x^2+3x) \phantom{+0} \\
\hline
-4x^2-3x+0 \\
-(-4x^2-16x-12) \\
\hline
13x+12
\end{array}$$

BOOK WANTS

$$\boxed{x-4+\frac{A}{x+3}+\frac{B}{x+1} = \frac{13x+12}{(x+3)(x+1)} \quad * \ x-4}$$

This gives $\frac{13x+12}{(x+3)(x+1)}$ to work with:

$$13x+12 = A(x+1) + B(x+3)$$

$x=-1$: $-1 = 2B \Rightarrow B = -\frac{1}{2}$

$x=-3$: $-27 = -2A \Rightarrow A = \frac{27}{2}$, so we now have

$$x-4+\frac{\frac{27}{2}}{x+3}-\frac{\frac{1}{2}}{x+1}$$

checked with Computer.

Check: $\cancel{\frac{(x-4)(x^2+4x+3) + (x+1) - 5(x+3)}{(x+3)(x+1)} = x^3}$

$\cancel{= \frac{x^3+4x^2+3x-4x^2-16x-12+13x+18-5x-15}{(x+3)(x+1)}} =$

202 §8.4 #s 4, 9, 20, 21, 27, 34, 40, 45, 46, 48

(4b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2} =$

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

#s 7-38 Evaluate the integral

(9) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$x-9 = A(x-2) + B(x+5)$

$x=2$: $-7 = 3B \Rightarrow B = -\frac{3}{7}$

$x=-5$: $-14 = -7A \Rightarrow A = 2$

$$= \int \frac{2\,dx}{x+5} - \frac{3}{7}\int \frac{dx}{x-2} = \boxed{2\ln|x+5| - \frac{3}{7}\ln|x-2| + C}$$

202 §8.4 #5 ~~8~~, 20, 21, 27, 34, 40, 45, 46, 48

20 $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2-5x+16 = A(x-2)^2 + B(x-2)(2x+1) + C(2x+1)$$

Matthew's suggestion: $x=2$: $10 = 5C \Rightarrow \boxed{C=2}$ (Not zero, as in class)

$x=1$: $12 = A - 3B + 3C$

$x=0$: $16 = 4A - 2B + C$

Use $c=2$:

$A-3B+6=12 \Longrightarrow A-3B=6$

$4A-2B+2=16 \longrightarrow 4A-2B=14$

$A=3B+6 \longrightarrow 4(3B+6)-2B=14$

$12B+24-2B=14$

$10B=-10$

$\boxed{B=-1}$

$A-3(-1)=6$

$A+4=6$

$\boxed{A=2}$

This gives

$$\int\left(\frac{2}{2x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}\right)dx$$

$$= \boxed{2\ln|2x+1| - \ln|x-2| - \frac{2}{(x-2)^1} + C}$$

202 §8.4 #s 21, 27, 34, 40, 45, 46, 48

(21) $$\int \frac{x^3+4}{x^2+4}\,dx = \int x\,dx - \int \frac{4x-4}{x^2+4}\,dx$$

$$\begin{array}{r|l} & x \\ \hline x^2+4 & x^3+0x^2+0x+4 \\ & -(x^3 \qquad +4x) \\ \hline & \qquad\quad -4x+4 \end{array}$$

$$= \tfrac{1}{2}x^2 - 2\int \frac{2x\,dx}{x^2+4} + 4\int \frac{1}{x^2+2^2}\,dx$$

$$= \boxed{\tfrac{1}{2}x^2 - 2\ln|x^2+4| + 4\cdot\tfrac{1}{2}\arctan\left(\tfrac{x}{2}\right) + C}$$

(27) $$\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx$$

$$\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$x^3+x^2+2x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$x=0$: $1 = 2B + D = 1$

$x=1$: $5 = (A+B)(3) + (C+D)(2) = 3A+3B+2C+2D = 5$

$x=-1$: $-1 = (-A+B)(3) + (-C+D)(2) = -3A+3B-2C+2D = -1$

$x=2$: $17 = (2A+B)(6) + (2C+D)(5) = 12A+6B+10C+5D = 17$

202 §8.4 #s 27, 34, 40, 45, 46, 48

27 cont'd

① $\boxed{2B + D = 1}$

② $3A + 3B + 2C + 2D = 5$

③ $-3A + 3B - 2C + 2D = -1$

④ $12A + 6B + 12C + 6D = 17$

② + ③

$6B + 4D = 4 \longrightarrow$

⑤ $\boxed{3B + 2D = 2}$

① $\longrightarrow D = 1 - 2B$, so ⑤ $3B + 2(1 - 2B) = 2$

$3B + 2 - 4B = 2$

$-B = 0$

$\boxed{B = 0} \longrightarrow \boxed{D = 1}$

∴ ② $3A + 2C + 2 = 5 \longrightarrow$ ⑥ $3A + 2C = 3$

③ $-3A - 2C + 2 = -1 \longrightarrow -3A - 2C = -3$ No Help

④ $12A + 12C + 6 = 17$

⑦ $12A + 12C = 11$

$3A = 3 - 2C$

$A = \dfrac{3 - 2C}{3}$

$12\left(\dfrac{3 - 2C}{3}\right) + 12C = 11$

$4(3 - 2C) + 12C = 11$

$12 - 8C + 12C = 11$

$4C = -1$

$C = -\dfrac{1}{4}$

202 §8.4 #s 34, 40, 45, 46, 48

$x^3 + x^2 + 2x + 1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$

$x^3 + x^2 + 2x + 1 = Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Cx + Dx^2 + D$

$x^3 = Ax^3 + Cx^3 \Longrightarrow 1 = A + C = 1$ ①

$x^2 = Bx^2 + Dx^2 \Longrightarrow 1 = B + D = 1$ ②

$2x = 2Ax + Cx \Longrightarrow 2 = 2A + C = 2$ ③

$1 = 2B + D \longrightarrow 1 = 2B + D = 1$ ④

② − ④ : $-B = 0 \longrightarrow \boxed{B = 0}$

$\longrightarrow \boxed{D = 1}$ from ②, directly

① − ③ : $-A = -1 \Longrightarrow \boxed{A = 1}$

$\Longrightarrow \boxed{C = 0}$ from ①, directly

This gives us

$$\int \frac{Ax+B}{x^2+1}\,dx + \int \frac{Cx+D}{x^2+2}\,dx = \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+2}$$

$$= \boxed{\frac{1}{2}\ln|x^2+1| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C}$$

202 §8.4 #s 34, 40, 45, 46, 48

(34) $\int \frac{x^3}{x^3+1}\,dx = \int 1\,dx - \int \frac{1}{x^3+1}\,dx$

$$\begin{array}{r|l} & 1 \\ x^3+1 & x^3+0x^2+0x+0 \\ & -(x^3 \qquad\qquad +1) \\ \hline & \qquad\qquad -1 \end{array}$$

FACTOR $x^3+1$ : $x=-1$ is a root :

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

So $(x+1)(x^2-x+1) = x^3+1$

$b^2-4ac = (-1)^2-4(1)(1) = -3 < 0$

Irreducible

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$\Rightarrow 1 = A(x^2-x+1) + (Bx+C)(x+1)$

$= Ax^2 - Ax + A + Bx^2 + Bx + Cx + C = 1 \Rightarrow$

(1) $Ax^2 + Bx^2 = 0x^2 \Rightarrow A+B=0$

(2) $-Ax + Bx + Cx = 0x \Rightarrow -A+B+C=0$

(3) $A+C=1$

(1) Says $A=-B \Rightarrow$ (2) says $-(-B)+B+C=0 \Rightarrow$

$2B+C=0 \rightarrow C=-2B$

and (3) now says $-B+C=1$

SUBTRACT THESE : $3B=-1 \Rightarrow \boxed{B=-\frac{1}{3}}$

$\Rightarrow C = \frac{2}{3}$

$\Rightarrow A = \frac{1}{3}$

202 §8.4 #s 34, 40, 45, 46, 48

This gives us

$$\cancel{\frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx}$$

$$\int dx - \int \frac{1}{x^3+1}\,dx = \int dx - \frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x-4}{x^2-x+1}\,dx$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{6}\int \frac{3\,dx}{x^2-x+1}$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| - \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$\left(x^2 - x + \left(\tfrac{1}{2}\right)^2 - \tfrac{1}{4} + 1 = \left(x-\tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right)$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}}{2}\left(x-\frac{1}{2}\right)\right) + C$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{4}(2x-1)\right) + C$$

202 S'08 8.4 #s 40, 45, 46, 48

#s 39-50 Make a substitution to turn the integrand into a rational function and then evaluate

40 $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

Let $u=\sqrt{x+3}$

Then $u^2=x+3$ & so

$x=u^2-3 \Rightarrow$

$dx=2u\,du$

$$= \int \frac{2u\,du}{2u+u^2-3}$$

$$= \int \frac{2u\,du}{u^2+2u-3} = \int \frac{2u\,du}{(u+3)(u-1)}$$

Scratch:

$$\frac{2u}{(u+3)(u-1)} = \frac{A}{u+3} + \frac{B}{u-1} \Rightarrow$$

$$2u = A(u-1)+B(u+3) = Au-A+Bu+3B$$

$$\Rightarrow 2u = Au+Bu \Rightarrow A+B=2$$

$$0=-A+3B \Rightarrow -A+3B=0 \Rightarrow A=3B$$

$$\Rightarrow A+B = 3B+B = 2 \Rightarrow \boxed{B=\tfrac{1}{2}} \Rightarrow A=\tfrac{3}{2}$$

This gives

$$\frac{3}{2}\int\frac{du}{u+3} + \frac{1}{2}\int\frac{du}{u-1} = \frac{3}{2}\ln|u+3| + \frac{1}{2}\ln|u-1| + C$$

$$= \boxed{\frac{3}{2}\ln\left|\sqrt{x+3}+3\right| + \frac{1}{2}\ln\left|\sqrt{x+3}-1\right| + C}$$

202 §8.4 #s 46, 48

46 $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

Let $u=\sqrt{x}$

$du=\frac{1}{2\sqrt{x}}dx$

$x=u^2$

$= \displaystyle\int \frac{\sqrt{1+u}}{u^2}\,2\sqrt{x}\,du$

$= \displaystyle\int \frac{\sqrt{1+u}\;2u\,du}{u^2} = 2\int \frac{\sqrt{1+u}}{u}\,du$

Let $v=1+u$

Then $dv=du$

$u=v-1$

$= 2\displaystyle\int \frac{\sqrt{v}\,dv}{v-1}$

Let $u=\sqrt{v} \Rightarrow du=\frac{1}{2\sqrt{v}}dv$

Then we have

$u^2=v$ and

$v-1=u^2-1$

$= 2\displaystyle\int \frac{\sqrt{v}\cdot 2\sqrt{v}}{u^2+1}\,du$

$= 2\displaystyle\int \frac{2u^2}{u^2+1}\,du = 4\int du - 4\int \frac{du}{u^2+1}$ missed it $= 4u - 4\arctan(u) + C$

$$\begin{array}{r} 1 \\ u^2+1 \overline{)\, u^2+0u+0} \\ -(u^2 \quad\quad +1) \\ \hline -1 \end{array}$$

$= 4\sqrt{v} - 4\arctan\sqrt{v} + C$

$= 4\sqrt{1+u} - 4\arctan\sqrt{u+1} + C$

$= \boxed{4\sqrt{1+\sqrt{x}} - 4\arctan\sqrt{\sqrt{x}+1} + C}$

$= 2\displaystyle\int \frac{2u^2}{u^2-1}\,du = 4\int \left(1+\frac{1}{u^2-1}\right)du$

$$\begin{array}{r} 1 \\ u^2-1 \overline{)\, u^2+0u+0} \\ -(u^2 \quad\quad -1) \\ \hline +1 \end{array}$$

$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1)+B(u-1) = Au+A+Bu-B = 1$

202 §8.4 #s 46, 48

46 cnt'd

$\Rightarrow Au + Bu = 0u \Rightarrow A + B = 0$

$A - B = 1$

$$\begin{array}{r} A + B = 0 \\ A - B = 1 \\ \hline 2A = 1 \end{array}$$

$A = \frac{1}{2} \Rightarrow B = -\frac{1}{2}$

This gives us

$$4\int du + 4\int \frac{1}{2}\left(\frac{1}{u-1}\right) du - 4\int \frac{1}{2}\left(\frac{1}{u+1}\right) du$$

$$= 4u + 2\ln|u-1| - 2\ln|u+1| + C$$

UN-SUBSTITUTING, STEP-BY-STEP

$$= 4\sqrt{v} + 2\ln|\sqrt{v} - 1| - 2\ln|\sqrt{v} + 1| + C$$

$$= 4\sqrt{u+1} + 2\ln|\sqrt{u+1} - 1| - 2\ln|\sqrt{u+1} + 1| + C$$

$$= 4\sqrt{\sqrt{x}+1} + 2\ln\left|\sqrt{\sqrt{x}+1} - 1\right| - 2\ln\left|\sqrt{\sqrt{x}+1} + 1\right| + C$$

48 $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x} dx$ let $u = \sin x \Rightarrow du = \cos x \, dx$

$$= \int \frac{du}{u^2 + u} = \int \frac{du}{u(u+1)}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} \Rightarrow 1 = A(u+1) + Bu = Au + A + Bu = 1$$

$$\Rightarrow A = 1 \text{ \& } A + B = 0 \Rightarrow B = -1$$

$$= \int \frac{du}{u} - \int \frac{du}{u+1} = \ln|u| - \ln|u+1| + C$$

$$= \boxed{\ln|\sin x| - \ln|\sin x + 1| + C}$$