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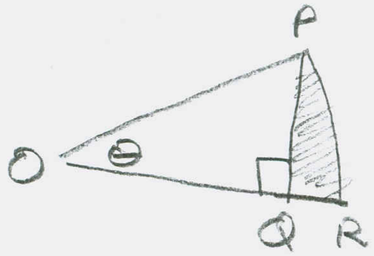
Area of the sector is $\frac{1}{2}r^2\theta$




Area is area of $\Delta OPQ$ plus the shaded area.

$\Delta OPQ$ : $\frac{1}{2} r\cos\theta \, r\sin\theta = \frac{1}{2}r^2\cos\theta\sin\theta$

Area of shaded region : $x^2+y^2=r^2 \Rightarrow y=\sqrt{r^2-x^2}$

$\Longrightarrow$ Area $= \displaystyle\int_{r\cos\theta}^{r} \sqrt{r^2-x^2}\,dx$ Let $x = r\cos u$ for $\theta \le u \le \frac{\pi}{2}$


Then $dx = -r\sin u \, du$ and the indefinite integral looks like $\displaystyle\int r\sin u \cdot (-r\sin u)\,du =$

$$-r^2\int \sin^2 u\,du = -r^2\int \tfrac{1}{2}(1-\cos(2u))\,du = -\frac{r^2}{2}\left[u - \tfrac{1}{2}\sin(2u)\right]+C$$

$$= -\frac{r^2}{2}u + \frac{r^2}{4}\sin(2u) + C = -\frac{r^2}{2}u + \frac{r^2}{2}\sin(u)\cos(u) + C$$

$$= -\frac{r^2}{2}\cos^{-1}\left(\frac{x}{r}\right) + \frac{1}{2} r\sin(u)\, r\cos(u) + C$$

$$= -\frac{r^2}{2}\cos^{-1}\left(\frac{x}{r}\right) + \frac{1}{2}\sqrt{r^2-x^2}\cdot x + C \Longrightarrow$$

$$\text{Area} = -\frac{r^2}{2}\left[\cos^{-1}\left(\frac{x}{r}\right)\right]_{r\cos\theta}^{r} + \frac{1}{2}\left[x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r}$$

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$$= -\frac{r^2}{2}\left[\cos^{-1}\left(\frac{r}{r}\right) - \cos^{-1}\left(\frac{r\cos\theta}{r}\right)\right]$$

$$+ \frac{1}{2}\left[r\sqrt{r^2-r^2} - r\cos\theta\sqrt{r^2-r^2\cos^2\theta}\right]$$

$$= -\frac{r^2}{2}\left[0 - \theta\right] + \frac{1}{2}\left[0 - r\cos\theta\, r\sin\theta\right]$$

$$= +\frac{r^2}{2}\theta - \frac{1}{2}r^2\cos\theta\sin\theta$$ = Shaded area.

Now add it to the area of the triangle:

$\triangle$ + shaded : $\frac{1}{2}r^2\cos\theta\sin\theta + \frac{1}{2}r^2\theta - \frac{1}{2}r^2\cos\theta\sin\theta$

$$= \frac{1}{2}r^2\theta$$

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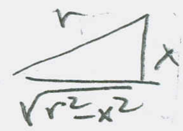
If we use the substitution




$x = r\sin u$ instead of $x = r\cos u$:

$dx = r\cos u\, du$ The indefinite integral becomes

$$\int r\cos u \cdot r\cos u\, du = r^2\int \cos^2 u\, du$$

$$= r^2\int \tfrac12(1+\cos(2u))\,du = \tfrac12 r^2\left[u+\tfrac12\sin(2u)\right]+C$$

$$= \tfrac12 r^2 u + \frac{r^2}{4}\sin(2u)+C = \tfrac12 r^2 u + \tfrac14 r^2(2\sin u\cos u)+C$$

$$= \tfrac12 r^2\sin^{-1}\left(\frac{x}{r}\right)+\tfrac12(r\sin u\; r\cos u)+C$$

$$= \tfrac12 r^2\sin^{-1}\left(\frac{x}{r}\right)+\tfrac12\left(x\sqrt{r^2-x^2}\right)+C \Longrightarrow \text{Area} =$$

$$\tfrac12 r^2\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_{r\cos\Theta}^{r} + \tfrac12\left[x\sqrt{r^2-x^2}\right]_{r\cos\Theta}^{r} =$$

$$\frac{r^2}{2}\left[\frac{\pi}{2}-\sin^{-1}(\cos\Theta)\right] + \tfrac12\left[0 - r^2\sin\Theta\cos\Theta\right]$$

This kills the $\Delta OPQ$ bit

$$\frac{r^2}{2}\left[\cos^{-1}(\cos\Theta)\right]$$

$$= \frac{r^2}{2}\Theta$$ (sine & cosine are $\frac{\pi}{2}$ out-of-phase.

$$\sin^{-1}(\cos\Theta) = \sin^{-1}\left(\sin\left(\Theta+\frac{\pi}{2}\right)\right) = \Theta+\frac{\pi}{2}\ !$$

The book way sidestepped this need for phase change