202 §7.2 BONUS

CLAIM

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall\, n \in \mathbb{N}$$

PROOF

$$1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1 \checkmark$$

Now, if $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, for some $k \geq 1$.

Then
$$\sum_{j=1}^{k+1} j^2 = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$
$$= \sum_{j=1}^{k} j^2 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)(k(2k+1)+6(k+1))}{6} = \frac{(k+1)(2k^2+k+6k+6)}{6}$$
$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \Rightarrow \text{it works}$$

for $n = k+1 \Rightarrow$ it works $\forall\, n \in \mathbb{N}$ by PMI ■

PMI = Principle of Mathematical Induction.