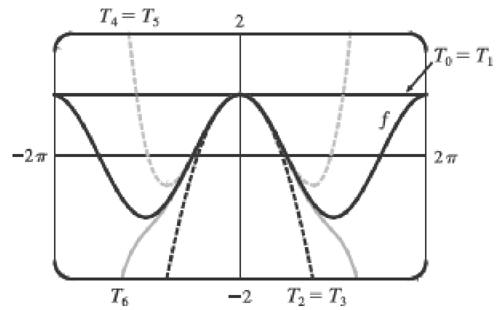
1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



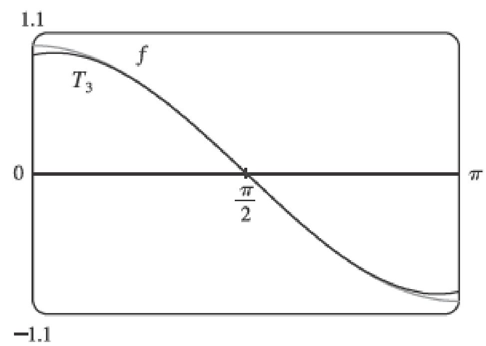
(b)

x	f	$T_0=T_1$	$T_2=T_3$	$T_4=T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

5.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1

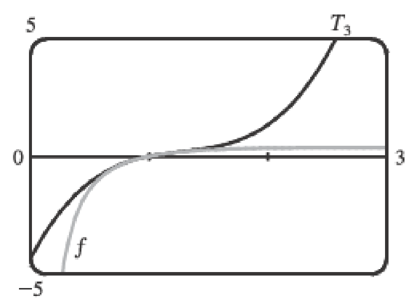


$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(\pi/2)}{n!}\left(x-\frac{\pi}{2}\right)^n$$

$$=-\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3$$

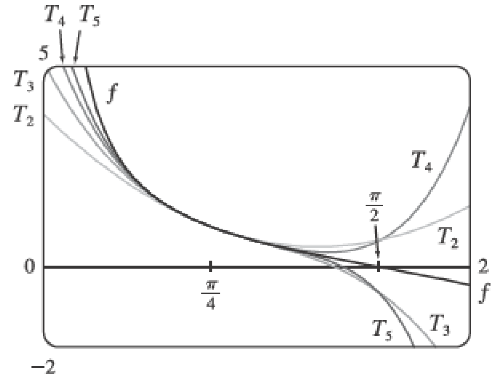
8.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\dfrac{\ln x}{x}$	0
1	$\dfrac{1-\ln x}{x^2}$	1
2	$\dfrac{-3+2\ln x}{x^3}$	-3
3	$\dfrac{11-6\ln x}{x^4}$	11



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(1)}{n!}(x-1)^n=(x-1)-\frac{3}{2}(x-1)^2+\frac{11}{6}(x-1)^3$$

11. You may be able to simply find the Taylor polynomials for $f(x) = \cot x$ using your CAS. We will list the values of $f^{(n)}(\pi/4)$ for $n = 0$ to $n = 5$.



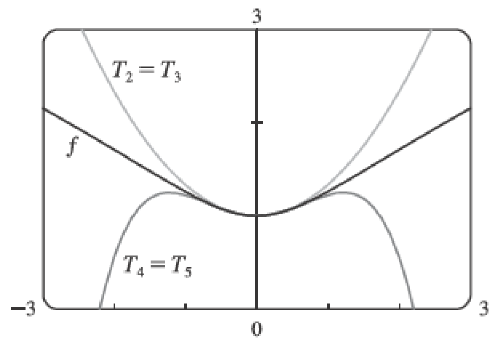
n	0	1	2	3	4	5
$f^{(n)}(\pi/4)$	1	−2	4	−16	80	−512

$$T_5(x) = \sum_{n=0}^{5} \frac{f^{(n)}(\pi/4)}{n!}\left(x - \tfrac{\pi}{4}\right)^n$$

$$= 1 - 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 - \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4 - \tfrac{64}{15}\left(x - \tfrac{\pi}{4}\right)^5$$

For $n = 2$ to $n = 5$, $T_n(x)$ is the polynomial consisting of all the terms up to and including the $\left(x - \frac{\pi}{4}\right)^n$ term.

12. You may be able to simply find the Taylor polynomials for $f(x) = \sqrt[3]{1 + x^2}$ using your CAS. We will list the values of $f^{(n)}(0)$ for $n = 0$ to $n = 5$.



n	0	1	2	3	4	5
$f^{(n)}(0)$	1	0	$\frac{2}{3}$	0	$-\frac{8}{3}$	0

$$T_5(x) = \sum_{n=0}^{5} \frac{f^{(n)}(0)}{n!}x^n = 1 + \tfrac{1}{3}x^2 - \tfrac{1}{9}x^4$$

For $n = 2$ to $n = 5$, $T_n(x)$ is the polynomial consisting of all the terms up to and including the x^n term. Note that $T_2 = T_3$ and $T_4 = T_5$.

16.

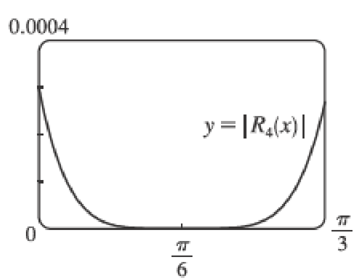
n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$1/2$
1	$\cos x$	$\sqrt{3}/2$
2	$-\sin x$	$-1/2$
3	$-\cos x$	$-\sqrt{3}/2$
4	$\sin x$	$1/2$
5	$\cos x$	

(a) $f(x) = \sin x \approx T_4(x)$

$$= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{6}\right) - \tfrac{1}{4}\left(x - \tfrac{\pi}{6}\right)^2 - \tfrac{\sqrt{3}}{12}\left(x - \tfrac{\pi}{6}\right)^3 + \tfrac{1}{48}\left(x - \tfrac{\pi}{6}\right)^4$$

(b) $|R_4(x)| \le \dfrac{M}{5!}\left|x - \frac{\pi}{6}\right|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{3} \;\Rightarrow\; -\frac{\pi}{6} \le x - \frac{\pi}{6} \le \frac{\pi}{6} \;\Rightarrow\; \left|x - \frac{\pi}{6}\right| \le \frac{\pi}{6} \;\Rightarrow\; \left|x - \frac{\pi}{6}\right|^5 \le \left(\frac{\pi}{6}\right)^5$. Since $\left|f^{(5)}(x)\right|$ is decreasing on $\left[0, \frac{\pi}{3}\right]$, we can take $M = \left|f^{(5)}(0)\right| = \cos 0 = 1$, so $|R_4(x)| \le \dfrac{1}{5!}\left(\dfrac{\pi}{6}\right)^5 \approx 0.000\,328$.

(c)



From the graph of $|R_4(x)| = |\sin x - T_4(x)|$, it seems that the error is less than $0.000\,297$ on $\left[0, \frac{\pi}{3}\right]$.

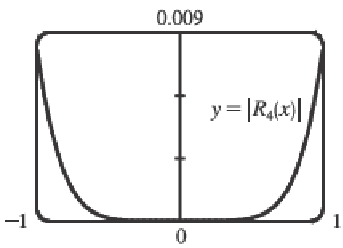
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x \sin x$	0
1	$\sin x + x \cos x$	0
2	$2\cos x - x \sin x$	2
3	$-3\sin x - x\cos x$	0
4	$-4\cos x + x \sin x$	-4
5	$5 \sin x + x \cos x$	

(a) $f(x) = x \sin x \approx T_4(x) = \dfrac{2}{2!}(x-0)^2 + \dfrac{-4}{4!}(x-0)^4 = x^2 - \dfrac{1}{6}x^4$

(b) $|R_4(x)| \le \dfrac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $-1 \le x \le 1 \;\Rightarrow\; |x| \le 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right| \le 5$ for $-1 \le x \le 1$. Thus, we can take $M = 5$ and get $|R_4(x)| \le \dfrac{5}{5!}\cdot 1^5 = \dfrac{1}{24} = 0.041\overline{6}$.

(c)



From the graph of $|R_4(x)| = |x \sin x - T_4(x)|$, it seems that the error is less than 0.0082 on $[-1, 1]$.

23. From Exercise 5, $\cos x = -\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3 + R_3(x)$, where $|R_3(x)| \le \dfrac{M}{4!}\left|x - \frac{\pi}{2}\right|^4$ with $\left|f^{(4)}(x)\right| = |\cos x| \le M = 1$. Now $x = 80° = (90° - 10°) = \left(\frac{\pi}{2} - \frac{\pi}{18}\right) = \frac{4\pi}{9}$ radians, so the error is $\left|R_3\left(\frac{4\pi}{9}\right)\right| \le \frac{1}{24}\left(\frac{\pi}{18}\right)^4 \approx 0.000\,039$, which means our estimate would *not* be accurate to five decimal places. However, $T_3 = T_4$, so we can use $\left|R_4\left(\frac{4\pi}{9}\right)\right| \le \frac{1}{120}\left(\frac{\pi}{18}\right)^5 \approx 0.000\,001$. Therefore, to five decimal places, $\cos 80° \approx -\left(-\frac{\pi}{18}\right) + \frac{1}{6}\left(-\frac{\pi}{18}\right)^3 \approx 0.17365$.

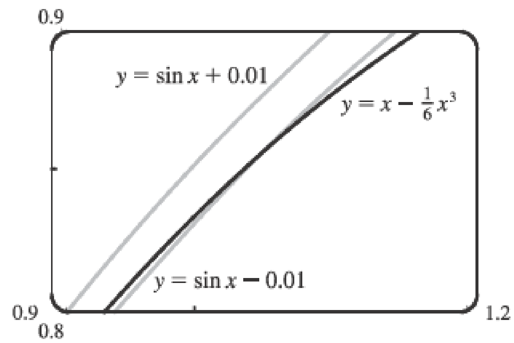
24. From Exercise 16, $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{6}\right)^4 + R_4(x)$, where $|R_4(x)| \le \dfrac{M}{5!}\left|x - \frac{\pi}{6}\right|^5$ with $\left|f^{(5)}(x)\right| = |\cos x| \le M = 1$. Now $x = 38° = (30° + 8°) = \left(\frac{\pi}{6} + \frac{2\pi}{45}\right)$ radians, so the error is $\left|R_4\left(\frac{38\pi}{180}\right)\right| \le \frac{1}{120}\left(\frac{2\pi}{45}\right)^5 \approx 0.000\,000\,44$, which means our estimate will be accurate to five decimal places. Therefore, to five decimal places, $\sin 38° = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{2\pi}{45}\right) - \frac{1}{4}\left(\frac{2\pi}{45}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{2\pi}{45}\right)^3 + \frac{1}{48}\left(\frac{2\pi}{45}\right)^4 \approx 0.61566$.

25. All derivatives of e^x are e^x, so $|R_n(x)| \le \dfrac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$, $R_n(0.1) \le \dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2$, and 3, we can estimate $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.1051\overline{6}$ and $e^{0.1} \approx 1.10517$.)

26. Example 6 in Section 11.9 gives the Maclaurin series for $\ln(1-x)$ as $-\sum_{n=1}^{\infty}\frac{x^n}{n}$ for $|x|<1$. Thus,

$\ln 1.4 = \ln[1-(-0.4)] = -\sum_{n=1}^{\infty}\frac{(-0.4)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (nonzero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \quad\Leftrightarrow$ $|x^5| < 120(0.01) \quad\Leftrightarrow\quad |x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x>0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.



30. $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty}\frac{(-1)^n\, n!}{3^n(n+1)\, n!}(x-4)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(n+1)}(x-4)^n$. Now

$f(5) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}(-1)^n b_n$ is the sum of an alternating series that satisfies (i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$, so by the Alternating Series Estimation Theorem, $|R_5(5)| = |f(5) - T_5(5)| \le b_6$, and

$b_6 = \frac{1}{3^6(7)} = \frac{1}{5103} \approx 0.000196 < 0.0002$; that is, the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.