4. $f(x) = \dfrac{3}{1-x^4} = 3\left(\dfrac{1}{1-x^4}\right) = 3(1+x^4+x^8+x^{12}+\cdots) = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty}3x^{4n}$

with $|x^4| < 1 \quad \Leftrightarrow \quad |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

[Note that $3\sum_{n=0}^{\infty}(x^4)^n$ converges $\quad \Leftrightarrow \quad \sum_{n=0}^{\infty}(x^4)^n$ converges, so the appropriate condition [from equation (1)] is $|x^4| < 1$.]

7. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n = \dfrac{x}{9}\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{9^{n+1}}$

The geometric series $\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\frac{x}{3}\right)^2\right| < 1 \quad \Leftrightarrow \quad \dfrac{|x^2|}{9} < 1 \quad \Leftrightarrow \quad |x|^2 < 9 \quad \Leftrightarrow \quad |x| < 3$, so

$R = 3$ and $I = (-3, 3)$.

10. $f(x) = \dfrac{x^2}{a^3 - x^3} = \dfrac{x^2}{a^3}\cdot\dfrac{1}{1-x^3/a^3} = \dfrac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\dfrac{x^3}{a^3}\right)^n = \sum_{n=0}^{\infty}\dfrac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \quad \Leftrightarrow$

$|x^3| < |a^3| \quad \Leftrightarrow \quad |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.

11. $f(x) = \dfrac{3}{x^2 - x - 2} = \dfrac{3}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} \quad \Rightarrow \quad 3 = A(x+1) + B(x-2)$. Let $x = 2$ to get $A = 1$ and

$x = -1$ to get $B = -1$. Thus

$$\frac{3}{x^2-x-2} = \frac{1}{x-2} - \frac{1}{x+1} = \frac{1}{-2}\left(\frac{1}{1-(x/2)}\right) - \frac{1}{1-(-x)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$

$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n - 1(-1)^n\right]x^n = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-2, 2)$ and the second converges for $(-1, 1)$.

Thus, the sum converges for $x \in (-1, 1) = I$.

13. (a) $f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) = -\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right] = -\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R = 1$.

14. (a) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$

[$C = 0$ since $f(0) = \ln 1 = 0$], with $R = 1$

(b) $f(x) = x\ln(1+x) = x\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n-1}$ with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

17. $\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n$ for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$. Now

$\frac{1}{(x-2)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n\right) = \sum_{n=1}^{\infty}\frac{n}{2^{n+1}}x^{n-1} = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n$. So

$f(x) = \frac{x^3}{(x-2)^2} = x^3\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^{n+3}$ or $\sum_{n=3}^{\infty}\frac{n-2}{2^{n-1}}x^n$ for $|x| < 2$. Thus, $R = 2$ and $I = (-2, 2)$.

18. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$. Thus,

$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty}(-1)^n\frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}x^{2n+1}$ for $\left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3$, so $R = 3$.

25. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with $R = 1$, so

$$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1} \text{ and}$$

$$\frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}, \text{ so}$$

$$\int \frac{x - \tan^{-1} x}{x^3}\, dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2 - 1}.$$ By Theorem 2, $R = 1$.

29. We substitute $3x$ for x in Example 7, and find that

$$\int x \arctan(3x)\, dx = \int x \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+2}}{2n+1}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+3}}{(2n+1)(2n+3)}$$

So
$$\int_0^{0.1} x \arctan(3x)\, dx = \left[\frac{3x^3}{1 \cdot 3} - \frac{3^3 x^5}{3 \cdot 5} + \frac{3^5 x^7}{5 \cdot 7} - \frac{3^7 x^9}{7 \cdot 9} + \cdots\right]_0^{0.1}$$
$$= \frac{1}{10^3} - \frac{9}{5 \times 10^5} + \frac{243}{35 \times 10^7} - \frac{2187}{63 \times 10^9} + \cdots.$$

The series is alternating, so if we use three terms, the error is at most $\frac{2187}{63 \times 10^9} \approx 3.5 \times 10^{-8}$. So

$$\int_0^{0.1} x \arctan(3x)\, dx \approx \frac{1}{10^3} - \frac{9}{5 \times 10^5} + \frac{243}{35 \times 10^7} \approx 0.000\,983$$ to six decimal places.

32. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$$ [substituting $n+1$ for n]

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad \Rightarrow \quad f''(x) + f(x) = 0.$$

35. (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 9.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.