1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.

   More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a, where a is a constant.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

   (i) 0 if the series converges only when $x = a$

   (ii) ∞ if the series converges for all x, or

   (iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

   In most cases, R can be found by using the Ratio Test.

   (b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

4. If $a_n = \dfrac{(-1)^n x^n}{n+1}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{n+2} \cdot \dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{1 + 1/(n+1)} = |x|$.

   By the Ratio Test, the series $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

7. If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{n+1}\right| = |x| \lim\limits_{n\to\infty} \dfrac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x.

   So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

10. If $a_n = \dfrac{10^n x^n}{n^3}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{10^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{10^n x^n}\right| = \lim_{n\to\infty} \left|\frac{10x\, n^3}{(n+1)^3}\right| = \lim_{n\to\infty} \frac{10\,|x|}{(1+1/n)^3} = \frac{10\,|x|}{1^3} = 10\,|x|$$

   By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{10^n x^n}{n^3}$ converges when $10\,|x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{10}$, so the radius of convergence is $R = \frac{1}{10}$.

   When $x = -\frac{1}{10}$, the series converges by the Alternating Series Test; when $x = \frac{1}{10}$, the series converges because it is a p-series with $p = 3 > 1$. Thus, the interval of convergence is $I = \left[-\frac{1}{10}, \frac{1}{10}\right]$.

13. If $a_n = (-1)^n \frac{x^n}{4^n \ln n}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)} \cdot \frac{4^n \ln n}{x^n}\right| = \frac{|x|}{4} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} \cdot 1$ [by l'Hospital's Rule] $= \frac{|x|}{4}$. By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \quad \Leftrightarrow \quad |x| < 4$, so $R = 4$. When $x = -4$, $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$. Since $\ln n < n$ for $n \geq 2$, $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$, $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

16. If $a_n = (-1)^n \frac{(x-3)^n}{2n+1}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{2n+3} \cdot \frac{2n+1}{(x-3)^n}\right| = |x-3| \lim_{n\to\infty} \frac{2n+1}{2n+3} = |x-3|$. By the Ratio Test, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$ converges when $|x-3| < 1 \quad [R = 1] \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$. When $x = 2$, the series $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverges by limit comparison with the harmonic series (or by the Integral Test); when $x = 4$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = (2, 4]$.

19. If $a_n = \frac{(x-2)^n}{n^n}$, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R = \infty$ and $I = (-\infty, \infty)$.

22. $a_n = \frac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)\,|x-4|^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{n\,|x-4|^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \frac{n^3+1}{n^3+3n^2+3n+2}\,|x-4| = |x-4|.$$

By the Ratio Test, the series converges when $|x-4| < 1 \quad [\text{so } R = 1] \quad \Leftrightarrow \quad -1 < x-4 < 1 \quad \Leftrightarrow \quad 3 < x < 5$. When $|x-4| = 1$, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$, which converges by comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2} \quad [p = 2 > 1]$. Thus, $I = [3, 5]$.

25. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{|4x+1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|4x+1|^n}\right] = \lim_{n\to\infty} \frac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio Test, the series converges when $|4x+1| < 1 \quad \Leftrightarrow \quad -1 < 4x+1 < 1 \quad \Leftrightarrow \quad -2 < 4x < 0 \quad \Leftrightarrow \quad -\frac{1}{2} < x < 0$, so $R = \frac{1}{4}$. When $x = -\frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x = 0$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series $[p = 2 > 1]$. $\quad I = \left[-\frac{1}{2}, 0\right]$.

28. If $a_n = \dfrac{n!\,x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\,x^n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x|}{2n+1} = \tfrac{1}{2}\,|x|.$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} a_n$ converges when $\frac{1}{2}\,|x| < 1 \quad\Rightarrow\quad |x| < 2$, so $R = 2$. When $x = \pm 2$,

$$|a_n| = \frac{n!\,2^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = \frac{[1 \cdot 2 \cdot 3 \cdot \cdots \cdot n]\,2^n}{[1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]} = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} > 1,$$ so both endpoint series

diverge by the Test for Divergence. Thus, the interval of convergence is $I = (-2, 2)$.

32. (a) Note that the four intervals in parts (a)–(d) have midpoint $m = \frac{1}{2}(p+q)$ and radius of convergence $r = \frac{1}{2}(q-p)$. We also know that the power series $\sum\limits_{n=0}^{\infty} x^n$ has interval of convergence $(-1, 1)$. To change the radius of convergence to r, we can change x^n to $\left(\frac{x}{r}\right)^n$. To shift the midpoint of the interval of convergence, we can replace x with $x - m$. Thus, a power series whose interval of convergence is (p, q) is $\sum\limits_{n=0}^{\infty}\left(\frac{x-m}{r}\right)^n$, where $m = \frac{1}{2}(p+q)$ and $r = \frac{1}{2}(q-p)$.

(b) Similar to Example 2, we know that $\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$ has interval of convergence $[-1, 1)$. By introducing the factor $(-1)^n$ in a_n, the interval of convergence changes to $(-1, 1]$. Now change the midpoint and radius as in part (a) to get $\sum\limits_{n=1}^{\infty}(-1)^n\,\frac{1}{n}\left(\frac{x-m}{r}\right)^n$ as a power series whose interval of convergence is $(p, q]$.

(c) As in part (b), $\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-m}{r}\right)^n$ is a power series whose interval of convergence is $[p, q)$.

(d) If we increase the exponent on n (to say, $n = 2$), in the power series in part (c), then when $x = q$, the power series $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{x-m}{r}\right)^n$ will converge by comparison to the p-series with $p = 2 > 1$, and the interval of convergence will be $[p, q]$.

33. No. If a power series is centered at a, its interval of convergence is symmetric about a. If a power series has an infinite radius of convergence, then its interval of convergence must be $(-\infty, \infty)$, not $[0, \infty)$.