1. $\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum\limits_{n=1}^{\infty} \frac{1}{n+3^n}$ converges by the Comparison Test.

3. $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{n}{n+2} = 1$, so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n \frac{n}{n+2}$ does not exist. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ diverges by the Test for Divergence.

5. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(n+1)^2\, 2^n}{(-5)^{n+1}} \cdot \frac{(-5)^n}{n^2\, 2^{n-1}}\right| = \lim\limits_{n\to\infty} \frac{2(n+1)^2}{5n^2} = \frac{2}{5} \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = \frac{2}{5}(1) = \frac{2}{5} < 1$, so the series $\sum\limits_{n=1}^{\infty} \frac{n^2\, 2^{n-1}}{(-5)^n}$ converges by the Ratio Test.

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\int \frac{1}{x\sqrt{\ln x}}\, dx \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\, du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim\limits_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim\limits_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim\limits_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

9. $\sum\limits_{k=1}^{\infty} k^2 e^{-k} = \sum\limits_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e}\right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1,$$ so the series converges.

11. $b_n = \frac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the given series $\sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

13. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}\right| = \lim\limits_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = 3 \lim\limits_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \frac{3^n n^2}{n!}$ converges by the Ratio Test.

15. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)[3(n+1)+2]} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{n!}\right| = \lim\limits_{n\to\infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1$,

so the series $\sum\limits_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$ converges by the Ratio Test.

17. $\lim\limits_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim\limits_{n\to\infty} (-1)^n\, 2^{1/n}$ does not exist and the series $\sum\limits_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/\left(2\sqrt{n}\right)} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n$. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

25. Use the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

27. $\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ [using integration by parts] $\overset{\text{H}}{=} 1$. So $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges by the Comparison Test.

29. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\cosh n} = \sum_{n=1}^{\infty} (-1)^n\, b_n$. Now $b_n = \frac{1}{\cosh n} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

Or: Write $\frac{1}{\cosh n} = \frac{2}{e^n + e^{-n}} < \frac{2}{e^n}$ and $\sum_{n=1}^{\infty} \frac{1}{e^n}$ is a convergent geometric series, so $\sum_{n=1}^{\infty} \frac{1}{\cosh n}$ is convergent by the Comparison Test. So $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\cosh n}$ is absolutely convergent and therefore convergent.

31. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{5^k}{3^k + 4^k} =$ [divide by 4^k] $\lim_{k\to\infty} \frac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim_{k\to\infty} \left(\frac{3}{4}\right)^k = 0$ and $\lim_{k\to\infty} \left(\frac{5}{4}\right)^k = \infty$.

Thus, $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

33. Let $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1}{n\sqrt{n}}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $[p = 3/2 > 1]$.