1. We plot T_0, T_2, T_4, and T_6 on the same graph as $\cos(x)$.

$with(plots):$

$$f0 := x \to \sum_{k=0}^{0} \frac{(-1)^k x^{2 \cdot k}}{(2 \cdot k)!}$$

$$x \to \sum_{k=0}^{0} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$f2 := x \to \sum_{k=0}^{1} \frac{(-1)^k x^{2 \cdot k}}{(2 \cdot k)!}$$

$$x \to \sum_{k=0}^{1} \frac{(-1)^k x^{2k}}{(2k)!}$$

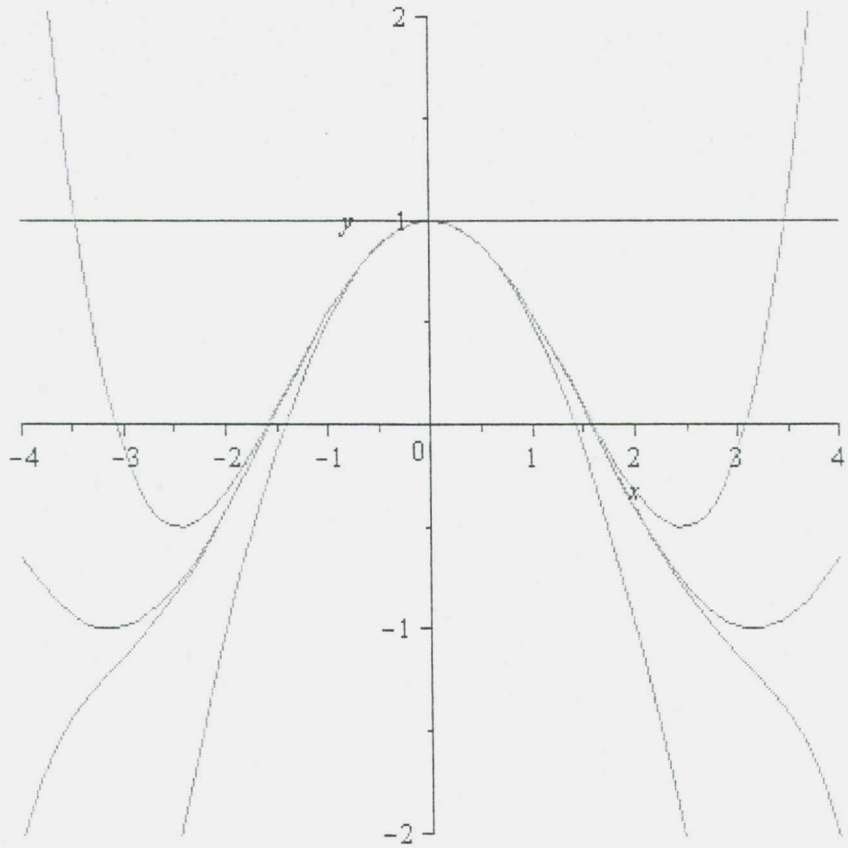
$$f4 := x \to \sum_{k=0}^{2} \frac{(-1)^k x^{2 \cdot k}}{(2 \cdot k)!}$$

$$x \to \sum_{k=0}^{2} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$f6 := x \to \sum_{k=0}^{3} \frac{(-1)^k x^{2 \cdot k}}{(2 \cdot k)!}$$

$$x \to \sum_{k=0}^{3} \frac{(-1)^k x^{2k}}{(2k)!}$$

$f0p := plot(f0(x), x = -4..4, y = -2..2) : \% :$
$f2p := plot(f2(x), x = -4..4, y = -2..2) : \% :$
$f4p := plot(f4(x), x = -4..4, y = -2..2) : \% :$
$f6p := plot(f6(x), x = -4..4, y = -2..2) : \% :$
$cosinep := plot(\cos(x), x = -4..4, y = -2..2) : \% :$
$display([f0p, f2p, f4p, f6p, cosinep])$



2
y
1
-4
-3
-2
-1
0
1
2
3
4
x
-1
-2

#s 3 - 10: Find the Taylor polynomial $T_n(x)$ for the function f at the number a. Graph f and T_3 on the same graph.

5. $f(x) = \cos(x)$, $a = \dfrac{\pi}{2}$

$f := x \to \cos(x)$

$$x \to \cos(x)$$

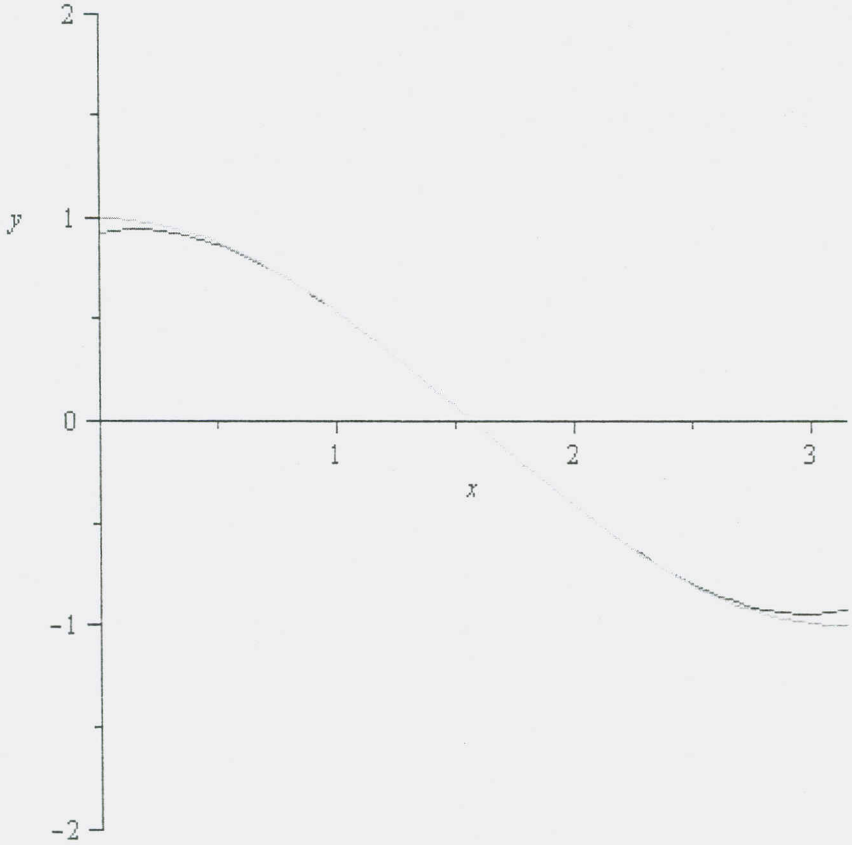
$t3 := taylor\left(f(x), x = \frac{\text{Pi}}{2}, 4\right)$

$$-\left(x - \frac{1}{2}\pi\right) + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3 + O\left(\left(x - \frac{1}{2}\pi\right)^4\right)$$

$t3 := x \to -\left(x - \frac{1}{2}\pi\right) + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3$

$$x \to -x + \frac{1}{2}\pi + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3$$

$plot([t3(x), f(x)], x = 0..\text{Pi}, y = -2..2)$



Very good match for values of x close to $x = 0$.

8. $f(x) = \dfrac{\ln(x)}{x}$

8. $f(x) = \dfrac{\ln(x)}{x}, a = 1$

$f := x \to \dfrac{\ln(x)}{x}$

$$x \to \frac{\ln(x)}{x}$$

$f1 := D(f)$

$$x \to \frac{1}{x^2} - \frac{\ln(x)}{x^2}$$

$f2 := D(f1)$

$$x \to -\frac{3}{x^3} + \frac{2\ln(x)}{x^3}$$

$f3 := D(f2)$

$$x \to \frac{11}{x^4} - \frac{6\ln(x)}{x^4}$$

$f4 := D(f3)$

$$x \to -\frac{50}{x^5} + \frac{24\ln(x)}{x^5}$$

$f(1)$

$$0$$

$f1(1)$

$$1$$

$f2(1)$

$$-3$$

$f3(1)$

$$11$$

$f4(1)$

$$-50$$

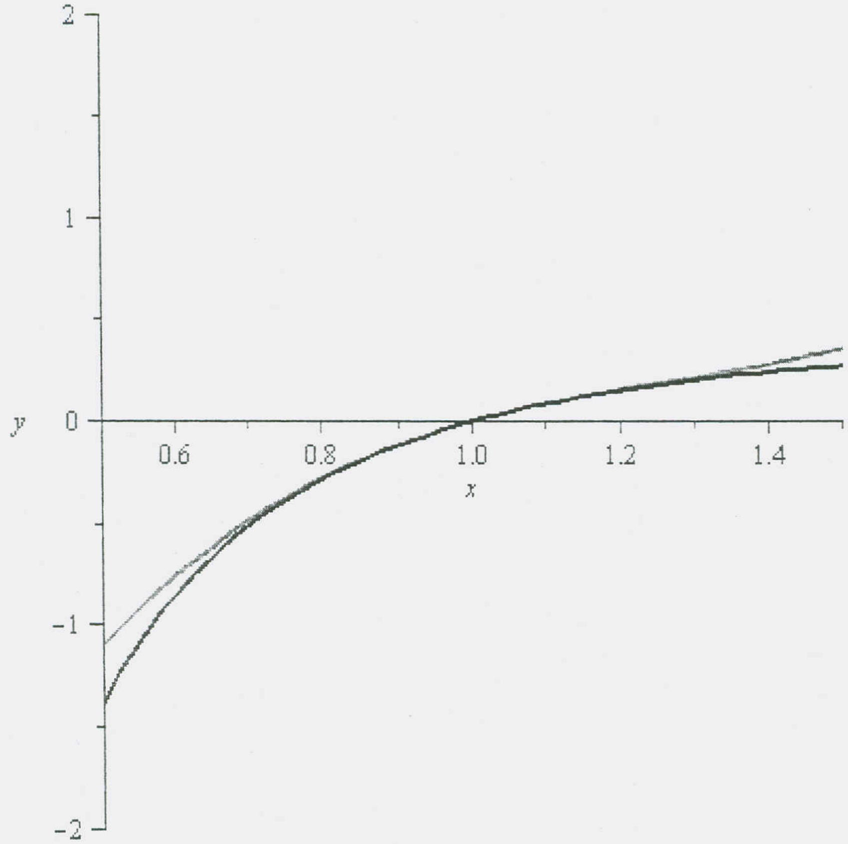
$T_3 := x \to 1 \cdot (x-1) - \dfrac{3}{2!} \cdot (x-1)^2 + \dfrac{11}{3!} \cdot (x-1)^3$

$$x \to x - 1 - \frac{3(x-1)^2}{2!} + \frac{11(x-1)^3}{3!}$$

$T3plot := plot\left(T_3(x), x = \frac{1}{2} .. \frac{3}{2}, y = -2 .. 2, color = red, thickness = 2\right) : \% :$

$fplot := plot\left(f(x), x = \frac{1}{2} .. \frac{3}{2}, y = -2 .. 2, color = black, thickness = 2\right) : \% :$

$display([T3plot, fplot])$



#s 11, 12: Use a computer algebra system to find the Taylor polynomials T_n *centered at a* **for** $n = 2, 3, 4, 5$. Then graph these polynomials and f on the same screen. I use a souped-up approach using the *taylor* command.

11. $f(x) = \cot(x),\ a = \frac{\pi}{4}$

$f := x \to \cot(x)$

$$x \to \cot(x)$$

$t2 := x \to taylor\left(f(x), x = \frac{Pi}{4}, 3\right)$

$$x \to taylor\left(f(x), x = \frac{1}{4}\pi, 3\right)$$

$t3 := x \to taylor\left(f(x), x = \frac{Pi}{4}, 4\right)$

$$x \to taylor\left(f(x), x = \frac{1}{4}\pi, 4\right)$$

$t4 := x \to taylor\left(f(x), x = \frac{Pi}{4}, 5\right)$

$$x \to taylor\left(f(x), x = \frac{1}{4}\pi, 5\right)$$

$t5 := x \to taylor\left(f(x), x = \frac{Pi}{4}, 6\right)$

$$x \to taylor\left(f(x), x = \frac{1}{4}\pi, 6\right)$$

Procedure to build the following polynomials: t3(x) enter, copy and paste the 3rd degree polynomial after typing $t3 := x \to$.

$t2 := x \to 1 - 2\left(x - \frac{1}{4}\pi\right) + 2\left(x - \frac{1}{4}\pi\right)^2$

$$x \to 1 - 2x + \frac{1}{2}\pi + 2\left(x - \frac{1}{4}\pi\right)^2$$

$t3 := x \to 1 - 2\left(x - \frac{1}{4}\pi\right) + 2\left(x - \frac{1}{4}\pi\right)^2 - \frac{8}{3}\left(x - \frac{1}{4}\pi\right)^3$

$$x \to 1 - 2x + \frac{1}{2}\pi + 2\left(x - \frac{1}{4}\pi\right)^2 - \frac{8}{3}\left(x - \frac{1}{4}\pi\right)^3$$

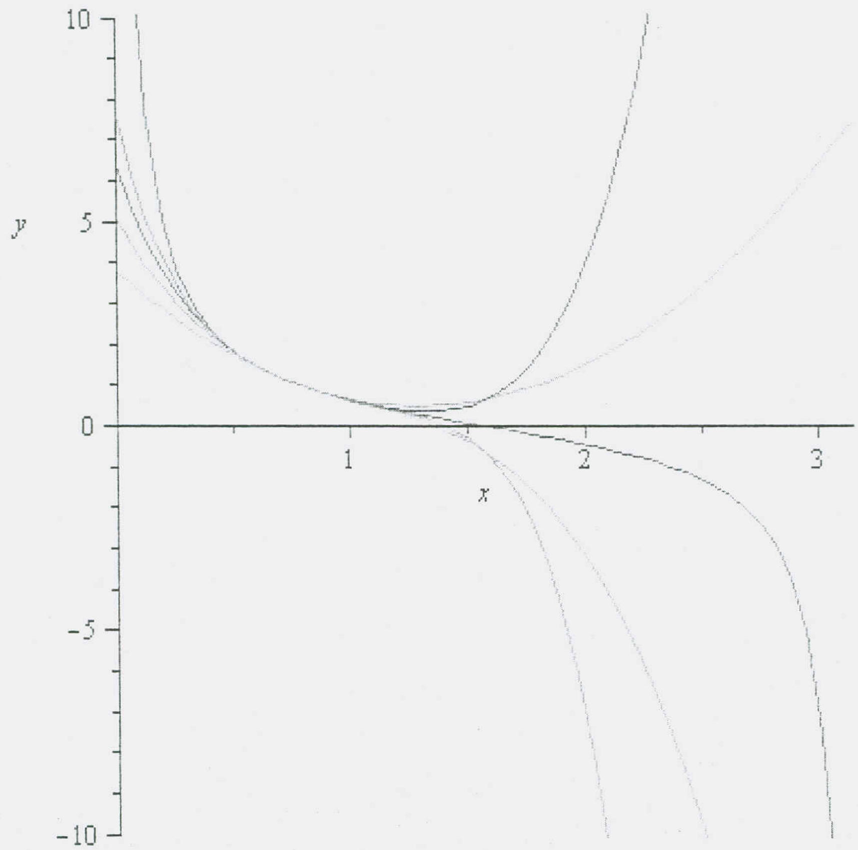
$t4 := x \to 1 - 2\left(x - \frac{1}{4}\pi\right) + 2\left(x - \frac{1}{4}\pi\right)^2 - \frac{8}{3}\left(x - \frac{1}{4}\pi\right)^3 + \frac{10}{3}\left(x - \frac{1}{4}\pi\right)^4$

$$x \to 1 - 2x + \frac{1}{2}\pi + 2\left(x - \frac{1}{4}\pi\right)^2 - \frac{8}{3}\left(x - \frac{1}{4}\pi\right)^3 + \frac{10}{3}\left(x - \frac{1}{4}\pi\right)^4$$

$$t5 := x \to 1 - 2\left(x - \frac{1}{4}\pi\right) + 2\left(x - \frac{1}{4}\pi\right)^2 - \frac{8}{3}\left(x - \frac{1}{4}\pi\right)^3 + \frac{10}{3}\left(x - \frac{1}{4}\pi\right)^4 - \frac{64}{15}\left(x - \frac{1}{4}\pi\right)^5$$

$$x \to 1 - 2x + \frac{1}{2}\pi + 2\left(x - \frac{1}{4}\pi\right)^2 - \frac{8}{3}\left(x - \frac{1}{4}\pi\right)^3 + \frac{10}{3}\left(x - \frac{1}{4}\pi\right)^4 - \frac{64}{15}\left(x - \frac{1}{4}\pi\right)^5$$

plot([cot(x), *t2*(x), *t3*(x), *t4*(x), *t5*(x)], x = 0 ..Pi, y =−10 ..10)



12. $f(x) = \sqrt[3]{1+x^2}$, $a = 0$.

$f := x \to \sqrt[3]{1+x^2}$

$$x \to (1+x^2)^{1/3}$$

$t2 := x \to taylor(f(x), x = 0, 3)$

$$x \to taylor(f(x), x = 0, 3)$$

$t3 := x \to taylor(f(x), x = 0, 4)$

$$x \to taylor(f(x), x = 0, 4)$$

$t4 := x \to taylor(f(x), x = 0, 5)$

$$x \to taylor(f(x), x = 0, 5)$$

$t5 := x \to taylor(f(x), x = 0, 6)$

$$x \to taylor(f(x), x = 0, 6)$$

$t2 := x \to 1 + \frac{1}{3} x^2$

$$x \to 1 + \frac{1}{3} x^2$$

$t3 := x \to 1 + \frac{1}{3} x^2$

$$x \to 1 + \frac{1}{3} x^2$$

$t4 := x \to 1 + \frac{1}{3} x^2 - \frac{1}{9} x^4$

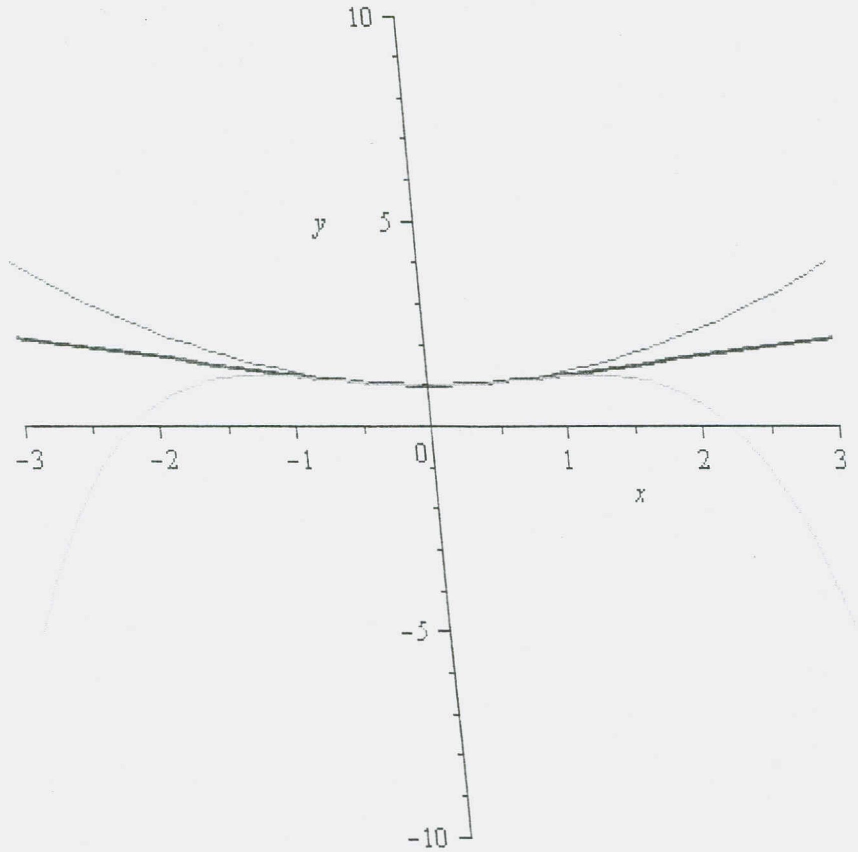
$$x \to 1 + \frac{1}{3} x^2 - \frac{1}{9} x^4$$

$t3, t5$ don't change anything.

$tplots := plot([t2(x), t4(x)], x = -3..3, y = -10..10) : \% :$

$fplot := plot(f(x), x = -3..3, y = -10..10, color = blue, thickness = 2) : \% :$

$display([fplot, tplots])$


10
5
y
-5
-10
-3
-2
-1
0
1
2
3
x

#s 13 - 22

(a) Approximate f by a Taylor plynomial with degree n at the number a.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation, when x lies in the given interval.

(c) Check your result in part (b) by graphing $|R_n(x)|$.

16. $f(x) = \sin(x)$, $a = \frac{\pi}{6}$, $n =$ ~~2~~, ~~$4 \le x \le 4.2$~~ → 4, $0 \le x \le \frac{\pi}{3}$

$f := x \to \sin(x)$

$$x \to \sin(x)$$

$t := x \to taylor\left(f(x), x = \frac{\text{Pi}}{6}, 3\right)$

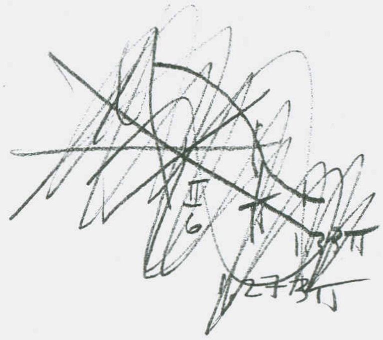
$$x \to taylor\left(f(x), x = \frac{1}{6}\pi, 3\right)$$

$t(x)$

$$\frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{1}{6}\pi\right) - \frac{1}{4}\left(x - \frac{1}{6}\pi\right)^2 + O\left(\left(x - \frac{1}{6}\pi\right)^3\right)$$

$t2 := x \to \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{1}{6}\pi\right) - \frac{1}{4}\left(x - \frac{1}{6}\pi\right)^2$

$$x \to \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{1}{6}\pi\right) - \frac{1}{4}\left(x - \frac{1}{6}\pi\right)^2$$

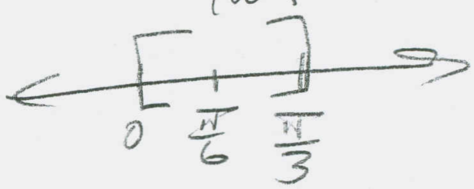


0	sin x
1	cos x
2	−sin x
3	−cos x
4	sin x
5	cos x

16 b

$$|R_4(x)| \le \frac{1}{5!}\left(\frac{\pi}{6}\right)^5 \approx$$

$[0, \frac{\pi}{3}]$



$\approx .000328$

21. $f(x) = x \cdot \sin(x)$, $a = 0$, $n = 4, -1 \le x \le 1$

$f := x \rightarrow x \cdot \sin(x)$

$$x \rightarrow x \sin(x)$$

$t := x \rightarrow taylor(f(x), x = 0, 5)$

$$x \rightarrow taylor(f(x), x = 0, 5)$$

$t(x)$

$$x^2 - \frac{1}{6}x^4 + O(x^6)$$

$t4 := x \rightarrow x^2 - \frac{1}{6}x^4$

$$x \rightarrow x^2 - \frac{1}{6}x^4$$

23.

23. Use #5 to approximate $\cos(80^0)$ correct to five decimal places. Have to execute the #5 stuff to get *f* and *t3* acting correctly.

$t3(x)$

$$-x + \frac{1}{2}\pi + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3$$

$t3\left(\frac{80 \cdot \mathrm{Pi}}{180}\right)$

$$\frac{1}{18}\pi - \frac{1}{34992}\pi^3$$

$evalf(\%)$

$$0.1736468291$$

So it looks like **0.17365** is correct to 5 places. I'm going to double check by looking at the next term in this alternating series:

$taylor\left(\cos(x), x = \frac{Pi}{2}, 6\right)$

$$-\left(x - \frac{1}{2}\pi\right) + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3 - \frac{1}{120}\left(x - \frac{1}{2}\pi\right)^5 + O\left(\left(x - \frac{1}{2}\pi\right)^6\right)$$

$t5 := x \to -\left(x - \frac{1}{2}\pi\right) + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3 - \frac{1}{120}\left(x - \frac{1}{2}\pi\right)^5$

$$x \to -x + \frac{1}{2}\pi + \frac{1}{6}\left(x - \frac{1}{2}\pi\right)^3 - \frac{1}{120}\left(x - \frac{1}{2}\pi\right)^5$$

$-\frac{1}{120}\left(\frac{80 \cdot Pi}{180} - \frac{1}{2}\pi\right)^5$

$$\frac{1}{226748160}\pi^5$$

evalf(%)

$$0.00000134960162$$

0.000001349601624+ 0.173646829(

$$0.173648178($$

Sure enough, it doesn't change in the 5th digit, when I add the next term in there. Good.

24.

#26

Use info from #16 to evaluate $\sin(38^0)$ correct to five decimal places.

$t2 := x \to \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{1}{6}\pi\right) - \frac{1}{4}\left(x - \frac{1}{6}\pi\right)^2$

$$x \to \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{1}{6}\pi\right) - \frac{1}{4}\left(x - \frac{1}{6}\pi\right)^2$$

$t2\left(\frac{38 \cdot Pi}{180}\right)$

$$\frac{1}{2} + \frac{1}{45}\sqrt{3}\,\pi - \frac{1}{2025}\pi^2$$

evalf(%)

$$0.616046079($$

From this, it appears that we have **0.61605** gives us the desired result.

Nope. Need to go to T_4 and then you get .61566

25.

Use Taylor's Inequality **to** *determine the n*

202 S12.11 #5, 25, 26, 27, 30

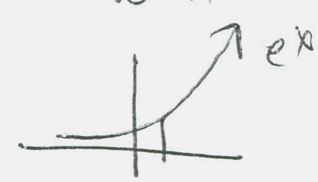
(25) We use Taylor's Inequality to determine the # of terms needed for the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001

Want $|R_n(x)| < .00001 = 10^{-5}$

Since $f^{(n)}(x) = e^x \ \forall n = 0, 1, 2, \ldots$ we find M for $|x-a| \le |.1 - 0| = .1$, which is to say, we maximize $f^{(n)}(x)$ on $[-.1, .1]$.

Since $\max_{x \in [-.1, .1]} \{|e^x|\} = e^{.1}$, define $M = e^{.1}$

This gives $|R_n(x)| \le \frac{e^{.1}}{(n+1)!} |.1|^{n+1}$ want $< .00001$



$|R_1(x)| \le \frac{e^{.1}}{2!}(.1)^2 \approx .005526$

$|R_2(x)| \le \frac{e^{.1}}{3!}(.1)^3 \approx 1.84195 \times 10^{-4} \approx .00018$

$|R_3(x)| \le \frac{e^{.1}}{4!}(.1)^4 \approx 4.6 \times 10^{-6} \approx .0000046$

$< .00001$

So T_3 will do it.

n=3

~~Something wrong here. Taylor's Inequality should be more conservative~~

Alternating Series Test gives the SAME result. Certainly no worse.

202 §12.11 #s 26, 27, 30

(26 cont'd) Wait! This is an alternating series.
and the 1st neglected term $|a_6| \approx .0007$
so T_5 does it. $\frac{(.1)^6}{6!} \approx$
$n = 5$ is the book answer.

I made some sort of boo-boo on
the Taylor's Inequality.

(27) What range of x-values will
keep $T_3(x) = x - \frac{x^3}{6} \approx \sin(x)$ within
.01, i.e., want $|R_n| < .01$.

$|R_3(x)| \leq \frac{M}{4!}|x|^4$, where
$M \geq |f^{(n)}(x)|$. Since $|f^{(n)}(x)| \leq 1$, $M \equiv 1$.
So, want $\frac{1}{4!}|x|^4 < .01 \Rightarrow$
$|x|^4 < (.01)(24) = .24 \Rightarrow$
$|x| < (.24)^{\frac{1}{4}} \approx .699927 1023$.
So, $-.69927 1023 \leq x \leq .69927 1023$.
WAIT! Alternating Series!

202 §12.11 #s 27, 30

(27) Using the easier estimate for Alternating series.

Notice that Taylor's Inequality is more conservative. Basically Alternating series converge relatively quickly

$\left|\frac{x^5}{5!}\right| < .01$

$|x^5| < 1.20$

$|x| < 1.037137$, approximately

$\Longrightarrow -1.037 \le x \le 1.037$ does it

(30) $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$ and the Taylor series converges to $f(x)$ $\forall$ x in the interval of convergence

Then the 5^{th} degree polynomial approximates $f(5)$ with error less than .0002.

$|x-a| = |5-4| = 1$

$|R_n(5)| \le \max_{x \in [4,5]} \left\{ \frac{f^{(6)}(x)}{6!} |1|^6 \right\}$

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n(n+1)} \frac{(x-4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$ is alternating

and with $x=5$, we have $f(5) = \sum_{n=0} \frac{(-1)^n}{3^n(n+1)}$

202 §12.11 #s 30

(30 cnt'd) since $\frac{1}{3^n(n+1)}$ is decreasing and

converges to zero, the error $|R_n(5)|$ is bdd

by $|a_6| = \frac{1}{3^6(7)} \approx 1.9596 \times 10^{-4} \approx .00019596$

and this is less than .0002 !