202 §12.10 I #s 4, 5, 10, 11, 13, 18, 22, 26, 34, 35

(4) Find the Taylor's Series, centered @ $a=4$, if $f^{(n)}(4) = \dfrac{(-1)^n n!}{3^n(n+1)}$. What is the radius of convergence?

Easy: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n n! (n+1)} (x-4)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{3^n (n+1)}}$

$R=3$

#s 5-12 Find the Maclaurin series. Also find R.

(5) $f(x) = (1-x)^{-2} = \displaystyle\sum_{n=0}^{\infty} \binom{-2}{n} x^n = 1 - 2(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3$

Using Binomial expansion

$-2-2+1$

$= \displaystyle\sum_{n=0}^{\infty} \frac{-2(-3)\cdots(-2-n+1)}{n!}(-1)^n x^n = \sum_{n=0}^{\infty} \frac{(-2)(-3)\cdots(-1-n)}{n!}(-1)^n x^n$

$R=1$

(10) $f(x) = xe^x = x\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}$

#10

$R=\infty$

I broke the rules on #5, but you might find it informative.

(11) $f(x) = \sinh(x) = \frac{1}{2}\left[e^x - e^{-x}\right]$

$= \displaystyle\frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}\right] = \boxed{\frac{1}{2}\sum_{n=0}^{\infty} \frac{(1-(-1)^n)}{n!} x^n} =$

$= \frac{1}{2}\left[0 + 2x + 0 + \frac{2x^3}{3!} + 0 + \frac{2x^5}{5!} + \cdots\right]$

$= \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

#s 13-20 Find Taylor series at $x=a$

(13) $f(x) = x^4 - 3x^2 + 1 \qquad a=1$

$f(1) = -1$

$f'(x$

202 §12.10 I #s 13, 18, 22, 26, 34, 35

13 I'll do it straight $a=1$

$f(x) = x^4 - 3x^2 + 1$ $\quad f(1) = -1$

$f'(x) = 4x^3 - 6x$ $\quad f'(1) = -2$

$f''(x) = 12x^2 - 6$ $\quad f''(1) = 6$

$f^{(3)}(x) = 24x$ $\quad f'''(1) = 24$

$f^{(4)}(x) = 24$ $\quad f^{(4)}(1) = 24$

$f^{(5)}(x) = 0$ $\quad f^{(5)}(1) = 0$

$$f(x) = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = -1 - 2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3$$

$$+ \frac{24}{4!}(x-1)^4 = \boxed{-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4}$$

18 $f(x) = \sin(x)$, $a = \frac{\pi}{2}$

#34

0	$\sin(x)$	$\sin\frac{\pi}{2} = 1$
1	$\cos(x)$	$\cos\frac{\pi}{2} = 0$
2	$-\sin(x)$	$-\sin(\frac{\pi}{2}) = -1$
3	$-\cos(x)$	$-\cos(\frac{\pi}{2}) = 0$
4	$\sin(x)$	$\sin(\frac{\pi}{2}) = 1$
5	$\cos(x)$	$\cos(\frac{\pi}{2}) = 0$

$$1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}}$$

$$\left| \frac{|x-\frac{\pi}{2}|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\frac{\pi}{2}|^{2n}} \right|$$

$$= \frac{|x-\frac{\pi}{2}|^2}{(2n+2)(2n+1)} \xrightarrow{n\to\infty} 0 \Rightarrow \boxed{R = \infty}$$

202 §12.10 I #s 22, 26, 34, 35

(22) Prove that your answer for #13 represents $\sin x$ for all x.

Taylor's Inequality: $|R_n(x)| \le \frac{M}{(n+1)!}|x-\frac{\pi}{2}|^{n+1}$

Since $f^{(n)}(x) = \pm \sin x$ or $\pm \cos x$, we know that $M = 1$ works $\forall n \in \mathbb{N}$, $\ni$

$$|R_n(x)| \le \frac{|x-\frac{\pi}{2}|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0 \Rightarrow$$ The series converges $\forall x$! :-)

More detail: once x is fixed, $x-\frac{\pi}{2}$ is just a real #, say $r = x - \frac{\pi}{2}$

since r is fixed, eventually, $n+1 > r$ and we just keep piling on the factors:

$(n+2)(n+1) \cdots$

$(n+3)(n+2)(n+1)$.

(26) Use the binomial series to expand. State radius of convergence.

~~26~~ $f(x) = \frac{1}{(1+x)} = \sum_{n=0}^{\infty} \binom{4}{n} x^n$ Newp!

$= 1 + 4x + \frac{4\cdot 3}{2!}x^2 + \frac{4\cdot 3\cdot 2}{3!}x^3 + \frac{4\cdot 3\cdot 2\cdot 1}{4!}x^4 + \cdots$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4\cdot 3 \cdots (4-(n+1)+1)x^{n+1}}{(n+1)!} \cdot \frac{n!}{((4)(3)\cdots(4-n+1)x^n}\right|$$

202 §12.10 I #s 26, 34, 35

26 cont'd

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{\infty} \binom{-4}{n} x^n$$

$$= 1 - 4x + \frac{(-4)(-5)}{2!} x^2 - \frac{(4)(5)(6)}{3!} x^3 + \frac{(4)(5)(6)(7)}{4!} x^4$$

$$- \frac{4(5)(6)(7)(8)}{5!} x^5 + \cdots + (-1)^n \frac{(4)(5)(6)(7)\cdots(n+3)}{n!} x^n + \cdots$$

$$\frac{(-1)^n \cancel{(2)(3)\cdots}(n+1)(n+2)(n+3)}{(2)(3)\cancel{n!}} x^n$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n}{6} (n+1)(n+2)(n+3) x^n}$$

Since $\left| \frac{2n+1}{2n} \right| \xrightarrow{n \to \infty} |x|$ $R = 1$

#s 29-38 Use a Maclaurin's series from Table 1

34 $f(x) = x^2 \tan^{-1}(x^3)$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}}$$

($R = 1$

Toughie!

35 $f(x) = \frac{x}{\sqrt{4+x^2}} = x \cdot \frac{1}{(4+x^2)^{\frac{1}{2}}} = x \cdot \frac{1}{(4(1+\frac{x^2}{4}))^{\frac{1}{2}}}$

$$= x \cdot \frac{1}{2} \cdot \frac{1}{(1+(\frac{x}{2})^2)^{\frac{1}{2}}} = \frac{x}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{((\frac{x}{2})^2)^n}{n!} = \frac{x}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n}}{2^{2n} n!}$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2^{2n+1} n!} = \frac{x}{2} + \frac{-\frac{1}{2} x^3}{2^3} + \frac{\frac{(-\frac{1}{2})(-\frac{3}{2}) x^5}{2^5}}{2!} + \frac{\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) x^7}{2^7}}{3!}$$

$$= \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!\, 2^{3n+1}} x^{2n+1}}$$