202 §12.9 #s 4, 7, 10, 11, 13, 14, 17, 18, 25, 29, 32, 35

#s 3-10 Find a power series representation

4) $f(x) = \frac{3}{1-x^4} = 3 \cdot \frac{1}{1-x^4} = 3\sum_{n=0}^{\infty} x^{4n} = \boxed{\sum_{n=0}^{\infty} 3x^{4n}}$

7) $f(x) = \frac{x}{9+x^2} = \frac{x}{9} \cdot \frac{1}{1+\left(\frac{x}{3}\right)^2} = \frac{x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n$

$= \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{3^n} = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{3^{n+2}}}$

IC: $\cancel{\left|\frac{x^{2n+3}}{3^{n+3}} \cdot \frac{3^{n+2}}{x^{2n+1}}\right|} = \left|\frac{x^2}{3}\right| < 1 \Rightarrow |x| < \sqrt{3}$

10) $f(x) = \frac{x^2}{a^3 - x^3} = \frac{x^2}{(a-x)(a^2+ax+x^2)}$

No. Wait. You CAN Part-frac, but don't need to!

$= \frac{x^2}{a^3\left(1-\frac{x^3}{a^3}\right)}$ ~~#~~14

$= \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\left(\frac{x}{a}\right)^3\right)^n = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\frac{x^{3n}}{a^{3n}} = \boxed{\sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}}$

R: $\left|\frac{x^{3n+5}}{a^{3n+6}} \cdot \frac{a^{3n+3}}{x^{3n+2}}\right| = \left|\frac{x^3}{a^3}\right| = \left|\frac{x}{a}\right|^3 < 1$

$\Longrightarrow |x|^3 < |a|^3 \Longrightarrow |x| < |a|$

$\boxed{R = |a|}$

$\sum \frac{a^{3n+2}}{a^{3n+3}} = \sum \frac{1}{a} \not\Longrightarrow$

$\sum \frac{(-a)^{3n+2}}{a^{3n+3}} = \sum -\frac{1}{a} \not\Longrightarrow$

$\boxed{IC = (-|a|, |a|)}$

202 §12.9 #s 11, 13, 14, 17, 18, 25, 29, 32, 35

#s 11, 12 Use partial fractions to build a power series.

11 $f(x) = \dfrac{3}{x^2-x-2} = \dfrac{3}{(x-2)(x+1)} = \dfrac{1}{x+2} - \dfrac{1}{x-1} =$

$$= \frac{1}{2(1+\frac{x}{2})} + \frac{1}{1-x}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n + \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty}\left((-1)^n\frac{x^n}{2^n} + x^n\right) = \boxed{\sum_{n=0}^{\infty}\left((-1)^n \cdot \frac{1}{2^n} + 1\right)x^n}$$

13 a Use differentiation to find a power series representation for $f(x) = \dfrac{1}{(1+x)^2}$

Observe $f(x) = g'(x)$ for $g(x) = -\dfrac{1}{1+x}$

$$= -\sum_{n=0}^{\infty}(-x)^n = -\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^{n+1}x^n$$

$$= -1 + x - x^2 + x^3 - x^4 + \cdots = g(x) \Longrightarrow$$

$$g'(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots = \boxed{\sum_{n=0}^{\infty}(-1)^n(n+1)x^n = f(x)}$$

$$\boxed{R=1}$$

202 §12.9 #s 13, 14, 17, 18, 29, 29, 32, 35

13 b Use a to find power series for

$h(x) = \frac{1}{(1+x)^3}$. $\boxed{R=1}$

By part a $\frac{1}{(1+x)^2} = f(x) = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

and $h(x) = \frac{1}{(1+x)^3} = \frac{d}{dx}\left[-\frac{1}{2}\frac{1}{(1+x)^2}\right] = \frac{d}{dx}\left[-\frac{1}{2}f(x)\right]$

$= \frac{d}{dx}\left[-\frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n\right] = \frac{d}{dx}\left[-\frac{1}{2}\left(1-2x+3x^2-4x^3+\ldots\right)\right]$

$= -\frac{1}{2}\left[-2+6x-12x^2+\ldots\right] = 1-3x+6x^2$

$= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n-1}(n+1)(n+2)x^n = \boxed{\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)x^n \quad \boxed{R=1}}$

c Use b to find power series for

$j(x) = \frac{x^2}{(1+x)^3}$; Multiply b by x^2 :

$\boxed{\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)x^{n+2}}$ $\boxed{R=1}$

202 §12.9 #5 14, 17, 18, 25, 29, 32, 35

14 Power Series for and radius of convergence of

a $f(x)=\ln(1+x) \Longrightarrow f'(x)=\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$

#18

$\Longrightarrow f(x)=f(0)+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}=f(x) \quad \boxed{R=1}}$

b Power series for $f(x)=x\ln(1+x)$:

$\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+2}}{n+1}}$ $\boxed{R=1}$

c Power series for $f(x)=\ln(x^2+1)$

$\boxed{f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1}}$ $\boxed{R=1}$

#s 15-18 Find Power Series for & give Radius

12 $f(x)=\frac{x^3}{(x-2)^2}$. Let $g(x)=\frac{1}{2-x}$. Then $\frac{1}{(x-2)^2}=\frac{d}{dx}[g(x)]$

$\boxed{R=2}$

$g(x)=\frac{1}{2-x}=\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}=\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$

$\Longrightarrow g'(x)=\sum_{n=1}^{\infty}\frac{n x^{n-1}}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{(n+1)x^n}{2^{n+2}} \Longrightarrow$

$f(x)=\sum_{n=0}^{\infty}\frac{(n+1)x^{n+3}}{2^{n+2}}$

202 §12.8 #s 18, 25, 29, 32, 35

18 $f(x) = \arctan\left(\frac{x}{3}\right)$

From E7, $g(x) = \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \Rightarrow$

$$\arctan\left(\frac{x}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1}(2n+1)}}$$

For $\left|\frac{x}{3}\right| < 1 \longrightarrow |x| < 3$, so $\boxed{R = 3}$

#s 23-26 Evaluate as a power series

25 $\int \frac{x - \arctan(x)}{x^3}\, dx$

$$= \int \left(\frac{1}{x^2} - \frac{1}{x^3}\arctan(x)\right) dx$$

$$= -\frac{1}{x} - \int \frac{1}{x^3} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\, dx$$

$$= -\frac{1}{x} - \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n-2}}{2n+1}\, dx = -\frac{1}{x} - \int \left(\frac{x^{-2}}{1} - \frac{x^0}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \cdots\right) d$$

$$= -\frac{1}{x} - \left[\frac{x^{-1}}{-1} - \frac{1}{3}x + \frac{x^3}{3\cdot 5} - \frac{x^5}{5\cdot 7} + \cdots\right]$$

$$= -\frac{1}{x} + \frac{1}{x} + \frac{1}{3}x - \frac{x^3}{3\cdot 5} + \frac{x^5}{5\cdot 7} + \cdots$$

$$= \frac{1}{3}x - \frac{x^3}{3\cdot 5} + \frac{x^5}{5\cdot 7} + \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+3)} \text{ OR } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)(2n+1)}}$$

202 §12.9 #s 29, 32, 35

(29) Approximate to 6 places with power series.

$$\int_0^{0.1} x \arctan(3x)\,dx = \int_0^{0.1} x \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}\,dx$$

$$= \int_0^{0.1} x\left[3x - \frac{3^3x^3}{3} + \frac{3^5x^5}{5} - \frac{3^7x^7}{7} + \cdots\right]dx$$

$$= \int_0^{.1}\left(3x^2 - 3^2x^4 + \frac{3^5x^6}{5} - \frac{3^7x^8}{7} + \cdots\right)dx$$

$$= \left[x^3 - \frac{3^2x^5}{5} + \frac{3^5x^7}{5\cdot7} - \frac{3^7x^9}{7\cdot9} + \frac{3^9x^{11}}{9\cdot11} + \cdots\right]_0^{.1}$$

$$= \Bigg[\underbrace{\underbrace{\underbrace{\underbrace{\overset{.001}{.1^3} - \frac{3^2(.1)^5}{5}}_{.000982} + \frac{3^5(.1)^7}{5\cdot7}}_{.0009826942857} - \frac{3^7(.1)^9}{7\cdot9}}_{.0009826595 7}} + \frac{3^9(.1)^{11}}{9\cdot11} + \cdots\Bigg]$$

$$\boxed{.000983}$$

202 S' 12.9 #s 32, 35

32 $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ solves $y'' + y = 0$

Proof:

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

$$f'(x) = \frac{-2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \frac{10x^9}{10!} + \frac{12x^{11}}{12!} + \cdots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!}$$

$$\Rightarrow f''(x) = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \frac{x^{10}}{10!}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = -f(x),$$

so $f''(x) + f(x) = y'' + y = 0$!

35 Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ solves $f'(x) = f(x)$

$$f'(x) = \frac{d}{dx}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x) \checkmark$$

By Thm 9.4.2, the only sol'n to $f'(x) = f(x)$ is $f(x) = Ke^x$ & since $f(0) = 1$, $K = 1$