202 S^{n} 12.7 #s 1-33 ODDS

#s 1-38 Check for convergence

(1) $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$. Since $\frac{1}{n+3^n} < \frac{1}{3^n}\ \forall\ n=1,2,\ldots$ and $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges (Geometric), the series $\boxed{\text{converges}}$

(3) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+2}$ $\boxed{\text{diverges}}$, since $\frac{n}{n+2} \xrightarrow{n\to\infty} 1 \neq 0$.

(5) $$\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n} = \sum_{n=1}^{\infty}(-1)^n \frac{n^2 2^{n-1}}{5^n}$$

$$= \sum_{n=1}^{\infty}(-1)^n 2n^2 \cdot \left(\frac{2}{5}\right)^n$$

$$f(x) = 2x^2\left(\frac{2}{5}\right)^x \Longrightarrow f'(x) = 4x\left(\frac{2}{5}\right)^x + 2x^2\left(\ln\left(\frac{2}{5}\right)\right)\left(\frac{2}{5}\right)^x$$

$$= \left(\frac{2}{5}\right)^x x\left(4 + 2\ln\left(\frac{2}{5}\right)x\right) \overset{\text{SET}}{=} 0 \longrightarrow$$

$$2\ln\left(\frac{2}{5}\right)x = -4 \longrightarrow x = \frac{-4}{2\ln\left(\frac{2}{5}\right)} = \frac{2}{\ln\left(\frac{5}{2}\right)} > 0$$

So $\underline{f'(x) \text{ is eventually negative.}}$

Now, $$\lim_{n\to\infty}\left(2n^2\left(\frac{2}{5}\right)^n\right) = \lim_{n\to\infty} \frac{2n^2}{\left(\frac{5}{2}\right)^n} \overset{L'H}{=}$$

$$\lim_{n\to\infty} \frac{4n}{\ln\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{4}{\left(\ln\left(\frac{5}{2}\right)\right)^2\left(\frac{5}{2}\right)^n} = 0$$

$\Longrightarrow$ Converges, by Alternating Series Criteria.

202 §12.7 #s 7-33 ODDS

(7) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

Ratio? $\lim_{n\to\infty} \left| \frac{(n+1)\sqrt{\ln(n+1)}}{n\sqrt{\ln(n)}} \right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) \lim_{n\to\infty}\sqrt{\frac{\ln(n+1)}{\ln(n)}}$

$= \sqrt{\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n)}} \overset{L'H}{=} \sqrt{\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}} = \sqrt{\lim_{n\to\infty}\frac{n}{n+1}} = 1$

No help.

By §12.6 #34, we know that $\frac{\ln(n)^{\frac{1}{2}}}{n}$ is eventually decreasing and converges to zero. We apply the integral test.

$$\int_2^{\infty} \ln(x)^{\frac{1}{2}} \cdot \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\frac{2}{3}\ln(x)^{\frac{3}{2}}\right]_2^t = \infty$$

$\rightarrow$ $\boxed{\text{Diverges.}}$

(9) $\sum_{k=1}^{\infty} k^2 e^{-k}$ will $\boxed{\text{converge.}}$ Ratio Test:

$\left|\frac{(k+1)^2 e^{-(k+1)}}{k^2 e^{-k}}\right| = \left|\frac{(k+1)^2}{k^2} e^{-1}\right| \xrightarrow{n\to\infty} e^{-1} < 1$ ✓

(11) $\sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n\ln(n)}$. $\boxed{\text{Converges,}}$ by Alternating Series Test:

n & $\ln(n)$ are positive, increasing, so $\frac{1}{n\ln(n)}$ is positive & decreasing. Also $\lim a_n = 0$ ✓

202 §12.7 #s 13–33 ODDS

(13) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ I think it converges, b/c $n!$ is "bigger" than 3^n

Ratio: $\left| \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right| = \left| \frac{3}{n+1} \cdot \frac{(n+1)^2}{n^2} \right|$

$\xrightarrow{n\to\infty} \frac{3}{n+1} \xrightarrow{n\to\infty} 0 < 1 \Rightarrow$ $\boxed{\text{Converges}}$

(15) $\sum_{n=0}^{\infty} \frac{n!}{2\cdot5\cdot8\cdots(3n+2)}$

Ratio: $\left| \frac{(n+1)!}{2\cdot5\cdot8\cdots(3n+2)(3(n+1)+2)} \cdot \frac{2\cdot5\cdot8\cdots(3n+2)}{n!} \right|$

$= \left| \frac{n+1}{3n+5} \right| \xrightarrow{n\to\infty} \frac{1}{3} < 1 \Rightarrow$ $\boxed{\text{Converges.}}$

(17) $\sum_{n=1}^{\infty} (-1)^n \cdot 2^{\frac{1}{n}}$. Test for divergence:

$y = 2^{\frac{1}{n}} \Rightarrow \ln(y) = \frac{1}{n}\ln(2) \xrightarrow{n\to\infty} 0 \Rightarrow$

$\lim_{n\to\infty} (2^{\frac{1}{n}}) = e^0 = 1 \neq 0$ $\boxed{\text{Diverges}}$

202 § 12.7 #s 19–33 ODDS

19 $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$ Should converge. Let's see

$f(x) = \frac{\ln(x)}{x^{\frac{1}{2}}}$.

$\lim_{x\to\infty} f(x) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to\infty}\left(2\frac{x^{\frac{1}{2}}}{x}\right) = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$ ✓

$f'(x) = \frac{\frac{1}{x}\cdot x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\ln x}{x} = \frac{x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\ln(x)}{x}$

$= \frac{x^{-\frac{1}{2}}(1-\frac{1}{2}\ln(x))}{x}$ is eventually negative

So $\boxed{\text{Converges}}$, by Alternating Series Test

21 $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}2^{2n}}{n^n} = \sum_{n=1}^{\infty}\frac{2^{2n}}{n^n}$

ROOT TEST: $\sqrt[n]{\frac{2^{2n}}{n^n}} = \frac{2^{\frac{2n}{n}}}{n^{\frac{n}{n}}} = \frac{2^2}{n} \xrightarrow{n\to\infty} 0$

$\longrightarrow \boxed{\text{Converges}}$ (n^n grows FAST)

23 $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ KEN gave me this one.

Compare (in the limit) to $\sum \frac{1}{n}$:

$\lim_{n\to\infty}\left|\frac{\tan(\frac{1}{n})}{\frac{1}{n}}\right| \overset{L'H}{=} \lim_{n\to\infty}\left|\frac{-\frac{1}{n^2}\sec^2(\frac{1}{n})}{-\frac{1}{n^2}}\right| = \lim_{n\to\infty}\sec^2(\tfrac{1}{n}) = 1 \longrightarrow$ $\boxed{\text{Diverges}}$

202 §12.7 #s 25–33 ODDS

(25) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

~~Intuition says factorial's bigger than exponential~~ ~~$\boxed{\text{Diverges}}$~~

Test for Divergence:

$$\lim_{n\to\infty} \frac{n!}{e^{n^2}} = \lim_{n\to\infty} \frac{n(n-1)\cdots(2)(1)}{\underbrace{e^n \cdots\cdots e^n}_{n \text{ factors}}} = 0 \text{. oops!}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \lim_{n\to\infty} \frac{n+1}{e^{n^2+2n+1-n^2}} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0$$

$\Rightarrow$ $\boxed{\text{Converges}}$

(27) $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3} =$ Intuition "p-test convergent"

Note: $k^{\frac{1}{2}} > \ln(k)$, eventually, since

$k^{\frac{1}{2}} > \ln(k)$ and

$$\frac{k \cdot k^{\frac{1}{2}}}{(k+1)^3} = \frac{k^{\frac{3}{2}}}{k^3+3k^2+3k+1} < \frac{k^{\frac{3}{2}}}{k^3} = \frac{1}{k^{\frac{3}{2}}} \text{ and}$$

$\sum \frac{1}{k^{3/2}}$ passes P-test $\boxed{\text{Converges}}$

Racetrack Principle: $\sqrt{1} > \ln(1) = 0$

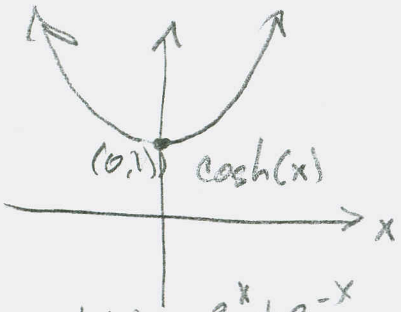
$$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} > \frac{1}{x}, \text{ eventually} \quad \frac{\sqrt{x}}{2} > 1$$

$\sqrt{x} > 2$

$\sqrt{4} = 2,\ \ln(4) \approx 1.38 \longrightarrow x > 4$

202 §12.7 #s 29-33 odds

(29) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh(n)}$ $\boxed{\text{Converges}}$ by Alternating Series Test.



$\cosh(x) = \frac{e^x + e^{-x}}{2}$

Since $\lim_{n \to \infty} \frac{1}{\cosh(n)} = 0$ and $\cosh(n)$ is increasing, $\forall n > 0$, we are done

(31) $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ Test for divergence

$$\frac{5^k}{3^k+4^k} > \frac{5^k}{4^k+4^k} = \frac{5^k}{2 \cdot 4^k} = \frac{1}{2}\left(\frac{5}{4}\right)^k \xrightarrow{k \to \infty} \infty$$

$\boxed{\text{Diverges}}$ Compare it to $\frac{1}{2}\left(\frac{5}{4}\right)^k$

(33) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$

compare to $\frac{1}{\sqrt{n}}$: $\left| \frac{\frac{\sin(\frac{1}{n})}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \right| = \left|\sin\left(\frac{1}{n}\right)\right| \xrightarrow{n \to \infty} 0$

Hmmm. "Smaller" than $\frac{1}{\sqrt{n}}$

Try $\frac{1}{n\sqrt{n}}$: $\frac{\frac{\sin(\frac{1}{n})}{\sqrt{n}}}{\frac{1}{n\sqrt{n}}} = \sin\left(\frac{1}{n}\right) \cdot n = \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$

$\xrightarrow{n \to \infty} 1$ So $\boxed{\text{converges}}$ by limit comparison to $\sum \frac{1}{n^{3/2}}$ p-test.