202 §12.4 #s 1–4, 12, 25, 26, 27, 31, 32, 35, 36, 38, 40, 41, 45, 46

(1) $\sum a_n, \sum b_n$ have positive terms and $\sum b_n \longrightarrow$

(a) If $a_n > b_n \ \forall n$, there's not much you can say about $\sum a_n$

(b) If $a_n < b_n \forall n$, then $\sum a_n \longrightarrow$.

(2) Same as #1, only $\sum b_n \not\longrightarrow$

(a) If $a_n > b_n$, then $\sum a_n \not\longrightarrow$

(b) If $a_n < b$, then so what?

~~(3)~~ #s 3 – 32 Determine if the series converges or diverges.

(3) $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{2n^2}$, since $a_n = \frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2}$, and we know $\frac{1}{2n^2} = \frac{1}{2} \cdot \frac{1}{n^2}$ and $\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} \longrightarrow$, by p-test.

(4) $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$. This one's easier to do with LIMIT COMPARISON to $\sum \frac{1}{n}$:

$a_n = \frac{n^3}{n^4-1}$, $b_n = \frac{1}{n}$. $\frac{a_n}{b_n} = \frac{n^3}{n^4-1} \cdot \frac{n}{1} = \frac{n^4}{n^4-1} \xrightarrow{n \to \infty} 1 > 0$

$\boxed{\sum a_n \text{ diverges}}$, since $\sum \frac{1}{n}$ diverges.

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(12) $\sum_{n=0}^{\infty} \frac{1+\sin(n)}{10^n}$. since $-1 \le \sin(n) \le 1$, we

see that $0 \le 1+\sin(n) \le 2$ and so

$\frac{1+\sin(n)}{10^n} \le \frac{2}{10^n} = 2 \cdot \frac{1}{10^n} = 2 \cdot \left(\frac{1}{10}\right)^n$ and we

Know that $\sum_{n=0}^{\infty} 2\left(\frac{1}{10}\right)^n = \sum_{n=1}^{\infty} 2 \cdot \frac{1}{10} \cdot \left(\frac{1}{10}\right)^{n-1}$ converges,

since it's a geometric series with $a = \frac{1}{5}$, $r = \frac{1}{10}$.

(25) $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ Let's Look @ the biggest stuff:

$$\frac{n^2}{\sqrt{n^6}} = \frac{n^2}{n^3} = \frac{1}{n}.$$

Limit Comparison to $\sum \frac{1}{n}$:

$$\frac{a_n}{b_n} = \frac{1+n+n^2}{\sqrt{1+n^2+n^6}} \cdot \frac{n}{1} = \frac{n^3+n^2+n}{\sqrt{n^6\left(\frac{1}{n^6}+\frac{1}{n^4}+1\right)}}$$

$$= \frac{n^3\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{n^3\sqrt{1+\frac{1}{n^6}+\frac{1}{n^4}}} = \frac{1+\frac{1}{n}+\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^6}+\frac{1}{n^4}}} \xrightarrow{n\to\infty} 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges

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(26) $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ BIG STUFF? $\frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{\frac{7}{3}}} = \frac{1}{n^{4/3}}$

#31 Limit comparison to

$$\frac{1}{n^{4/3}} : \frac{n+5}{\sqrt[3]{n^7+n^2}} \cdot \frac{n^{4/3}}{1} = \frac{a_n}{b_n} = \frac{n^{7/3}+5n^{4/3}}{n^{7/3}\sqrt[3]{1+\frac{1}{n^5}}}$$

$$= \frac{n^{\frac{7}{3}}(1+5n^{-1})}{n^{\frac{7}{3}}\sqrt[3]{1+\frac{1}{n^5}}} \xrightarrow{n \to \infty} \frac{1}{1} = 1 > 0 \Longrightarrow$$

$\boxed{\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \text{ converges}}$ p-test, comparison (in the limit) to $\sum \frac{1}{n^{4/3}}$

(27) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{e}\right)^n$

I'm thinking it'll compare nicely to $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$

$= \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)\left(\frac{1}{e}\right)^{n-1}$, $a \cdot r^{n-1}$, which converges: To make it easier

$$\frac{a_n}{b_n} = \frac{\left(\frac{n+1}{n}\right)^2 \left(\frac{1}{e}\right)^n}{\left(\frac{1}{e}\right)^n} = \left(\frac{n+1}{n}\right)^2 \xrightarrow{n \to \infty} 1 > 0, \text{ so}$$

$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ converges

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(31) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \sin(1) + \sin\left(\frac{1}{2}\right) + \cdots + \sin\left(\frac{1}{n}\right) + \cdots$

Hmmm. Recall $\frac{\sin(x)}{x} \xrightarrow{x \to 0} 1$. So let's have a look @ $\sum_{n=1}^{\infty} \frac{1}{n}$ & limit-compare:

$$\frac{a_n}{b_n} = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = n \sin\left(\frac{1}{n}\right).$$ Let $u = \frac{1}{n}$, then

$\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{u \to 0} \frac{1}{u} \sin(u) = 1$, so

$\boxed{\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \text{ diverges,}}$ since $\sum_{n=1}^{\infty} \frac{1}{n}$ does.

(32) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ Looks a lot like $\sum \frac{1}{n}$, to me.

#36

$$\frac{a_n}{b_n} = \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \frac{1}{n \cdot n^{1/n}} \cdot \frac{n}{1} = \frac{1}{n^{1/n}} \xrightarrow{n \to \infty} ?$$

Let's see what $\lim_{n \to \infty} n^{\frac{1}{n}}$ is:

$$y = n^{\frac{1}{n}} \to \ln(y) = \frac{1}{n} \ln(n) \xrightarrow[n \to \infty]{L'H} \frac{\frac{1}{n}}{1} = \frac{1}{n}$$

$\xrightarrow{n \to \infty} 0$. Thus $\lim_{n \to \infty} \ln(y) = 0 \Rightarrow$

$\lim_{n \to \infty} y = e^0 = 1$, so $n^{1/n} \xrightarrow{n \to \infty} 1$ & $\frac{1}{n^{1/n}} \longrightarrow 1$

So, $\boxed{\sum \frac{1}{n^{1+1/n}} \text{ diverges,}}$ since $\sum \frac{1}{n}$ diverges

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#s 33-36 Use the 1st ten terms to approximate the series. Estimate the error.

35 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2^n}$

$$\sum_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{1+2} + \frac{1}{1+4} + \frac{1}{1+8} + \frac{1}{1+16} + \frac{1}{1+32}$$

$$+ \frac{1}{1+64} + \frac{1}{1+128} + \frac{1}{1+256} + \frac{1}{1+512} + \frac{1}{1+1024}$$

$$= \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{33} + \frac{1}{65} + \frac{1}{129} + \frac{1}{257} +$$

$$\frac{1}{513} + \frac{1}{1025} = \frac{1078574620 7848}{14126278634325} \approx \boxed{.7635235356}$$

$a_n = \frac{1}{1+2^n} < \frac{1}{2^n}$, so

$$R_{10} = \sum_{n=11}^{\infty} \frac{1}{1+2^n} < \sum_{n=11}^{\infty} \frac{1}{2^n} = \cdots = \boxed{\frac{1}{512}} \approx .001953125,$$

ceiling on the error.

36 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$. Compare to $\sum \frac{1}{3^n}$, since

$$\frac{n}{n+1} \xrightarrow{n\to\infty} 1 \;:$$

$$\frac{a_n}{b_n} = \frac{n}{(n+1)3^n} \cdot \frac{3^n}{1} = \frac{n}{n+1} \xrightarrow{n\to\infty} 1, \text{ so it converges.}$$

$$\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{51578027}{181870920} \approx .2835968884.$$

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$$\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n} \quad \text{Compare to } \sum \frac{1}{3^n}:$$

$$\frac{a_n}{b_n} = \left(\frac{n}{n+1}\right)\left(\frac{1}{3^n}\right)\left(\frac{3^n}{n}\right) = \frac{n}{n+1} \xrightarrow{n \to \infty} 1 > 0$$

$\Rightarrow$ $\boxed{\text{Converges.}}$

Now, $\displaystyle\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \boxed{\frac{51578027}{181870920} \approx .2835968884 \approx S_{10}}$

Error Estimate: $\displaystyle R_n = R_{10} = \sum_{n=11}^{\infty} \frac{n}{(n+1)3^n} < \sum_{n=11}^{\infty} \frac{1}{3^n} =$

$$\sum_{n=11}^{\infty} \frac{1}{3^{11}} \cdot \frac{1}{3^{n-11}} = \sum_{n=11}^{\infty} \frac{1}{3^{11}} \left(\frac{1}{3}\right)^{n-11} = \sum_{n=0}^{\infty} \frac{1}{3^{11}} \left(\frac{1}{3}\right)^{n}$$

$$= \frac{1}{3^{11}} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{3^{11}} \cdot \frac{1}{\frac{2}{3}} = \frac{1}{3^{11}} \cdot \frac{3}{2} = \frac{1}{2 \cdot 3^{10}}$$

$$= \boxed{\frac{1}{118098} \approx .000008467543904 \approx R_{10}}$$

$$= 8.467543904 \times 10^{-6}$$

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36 cnt'd     Error estimate

$$R_n = \sum_{n=11}^{\infty} \frac{n}{(n+1)3^n} < \sum_{n=11}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^{n+10}} = \sum_{n=1}^{\infty} \left(\frac{1}{3^{11}}\right)\left(\frac{1}{3^{n-1}}\right)$$

$$= \frac{\frac{1}{3^{11}}}{1-\frac{1}{3}} = \frac{3}{2}\cdot\frac{1}{3^{11}} = \frac{1}{2\cdot 3^{10}} = \frac{1}{118098} \approx$$

$$\boxed{.00000846754390 4}$$ Error Estimate

38 For what values of p does $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$ converge?

Consider $\int_2^{\infty} \frac{1}{x^p \ln(x)}\,dx$

$w = \ln(x) \Rightarrow dw = \frac{1}{x}\,dx$

$$= \int_2^{\infty} \frac{\frac{1}{x}\,dx}{x^{p-1}\ln(x)} = \int \frac{dw}{w\, x^{p-1}} = \int w\, x^{1-p}\,dw$$

Hmmm. By parts $\frac{1}{\ln(x)}$ ~~dx~~ as u? Hmmm. Then $du = \frac{d}{dx}\left(\left[\ln(x)\right]^{-1}\right)$

$= -\ln(x)^{-2}\cdot\frac{1}{x}$ doesn't get rid of the ln(x)

Integral test would work, if the integral weren't so intractable. ~~Ca~~ So let's get down to cases?

202 §12.4 #s 38, 40, 41, 45, 46.

$p < 0$:

$\sum \frac{1}{n^p \ln(n)} > \sum \frac{1}{\ln(n)}$ which diverges, by previous work. ($\ln(n)$ grows slower than $n$)

$0 = p$: $\sum \frac{1}{\ln(n)}$ diverges

$0 < p < 1$:

$n^p < n \Rightarrow \sum \frac{1}{n^p \ln(n)} > \sum \frac{1}{n \ln(n)}$ diverges, by §12.3 #21. ($\int_2^\infty \frac{dx}{x \ln(x)}$ diverges)

so ~~p~~ $0 < p < 1$ is out.

$p = 1$: Same deal, since $\sum \frac{1}{n \ln(n)} \not\to$.

Now, $p > 1$: Use limit comparison, with

$\sum \frac{1}{n^p} = \sum b_n$: $\frac{a_n}{b_n} = \frac{\frac{1}{n^p \ln(n)}}{\frac{1}{n^p}} = \frac{1}{n^p \ln(n)} \cdot \frac{n^p}{1}$

$= \frac{1}{\ln(n)} \xrightarrow{n \to \infty} 0$. Since $\sum \frac{1}{n^p}$ converges for $p > 1$,

and $\frac{a_n}{b_n} \xrightarrow{n\to\infty} 0$, we see that $\sum \frac{1}{n^p \ln(n)}$ converges for $p > 1$. Direct comparison to $\sum \frac{1}{n^p}$ would work, too, since $\frac{1}{n^p \ln(n)} < \frac{1}{n^p}$ for $n \geq 3$ ($\ln(n) \geq 1$ $\forall n \geq 3$).

202 §12.4 #s 40, 41, 45, 46

40 $a_n, b_n > 0 \ \forall n \in \mathbb{N}$. $\sum b_n$ converges.

(a) Prove that if $\frac{a_n}{b_n} \xrightarrow{n\to\infty} 0$, then $\sum a_n$ converges.

$\boxed{\text{Pf}}$ $\frac{a_n}{b_n} \xrightarrow{n\to\infty} 0$ means that, given $\varepsilon > 0$,

$\exists N \in \mathbb{N} \ni \left|\frac{a_n}{b_n} - 0\right| < \varepsilon \ \forall n \geq N$.

This means $\frac{a_n}{b_n} < \varepsilon$ $n = N, N+1, \ldots$, i.e.,

$a_n < \varepsilon b_n < b_n$, if we assume $\varepsilon < 1$. Thus

$\sum a_n$ converges by comparing its N-TAIL to

$\sum_{n=N+1}^{\infty} b_n$. A bit sloppy, here, on N-vs-N+1.

But that's the gist. $\blacksquare$

(b) Use part (a) to show that the series converges:

(i) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$. Since $\ln(n) < n$ for all $n$,

we can compare this to $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$, which

converges, so $\sum \frac{\ln(n)}{n^3}$ converges.

(ii) $\sum \frac{\ln(n)}{\sqrt{n}\, e^n}$. Compare to $\sum \frac{1}{e^n}$:

$$\frac{\ln(n)}{\sqrt{n}\, e^n} \cdot \frac{e^n}{1} = \frac{\ln(n)}{\sqrt{n}} \xrightarrow[n\to\infty]{L'H} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \frac{2\sqrt{n}}{n} \xrightarrow{n\to\infty} 0,$$

so $\sum \frac{\ln(n)}{\sqrt{n}\, e^n}$ converges, because $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges.

202 §12.4 #s 41, 45, 46

(41)

(a) $a_n, b_n > 0 \ \forall n$, $\sum b_n$ does NOT converge.

Suppose that $\frac{a_n}{b_n} \xrightarrow{n \to \infty} \infty$. Then

$\sum a_n$ diverges

$\boxed{\text{Pf}}$ If $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \infty$, then $\frac{a_n}{b_n} > M$,

eventually, for any $M$ you could name. So,

in particular, $\exists N \in \mathbb{N} \ni$

$$\frac{a_n}{b_n} > M = 1 \quad \forall \ n > N \ \text{\& so,}$$

$a_n > b_n \ \forall n > N$, and $\sum a_n$ diverges,

by the (direct) Comparison Test. ▨

(b) Use (a) to show that the series

diverges.

(i) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$