1. $r=\theta^2,\ 0\le\theta\le\frac{\pi}{4}.\quad A=\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta=\int_0^{\pi/4}\frac{1}{2}(\theta^2)^2\,d\theta=\int_0^{\pi/4}\frac{1}{2}\theta^4\,d\theta=\left[\frac{1}{10}\theta^5\right]_0^{\pi/4}=\frac{1}{10}\left(\frac{\pi}{4}\right)^5=\frac{1}{10{,}240}\pi^5$

4. $r=\sqrt{\sin\theta},\ 0\le\theta\le\pi.\quad A=\int_0^{\pi}\frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 d\theta=\int_0^{\pi}\frac{1}{2}\sin\theta\,d\theta=\left[-\frac{1}{2}\cos\theta\right]_0^{\pi}=\frac{1}{2}+\frac{1}{2}=1$

7. $r=4+3\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$

$$A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}((4+3\sin\theta)^2\,d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(16+24\sin\theta+9\sin^2\theta)\,d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(16+9\sin^2\theta)\,d\theta \quad \text{[by Theorem 5.5.7(b)]}$$

$$=\frac{1}{2}\cdot 2\int_0^{\pi/2}\left[16+9\cdot\frac{1}{2}(1-\cos 2\theta)\right]d\theta \qquad \text{[by Theorem 5.5.7(a)]}$$

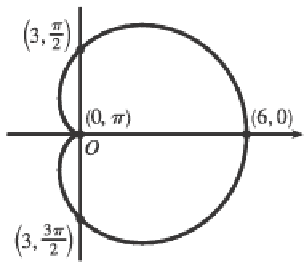
$$=\int_0^{\pi/2}\left(\frac{41}{2}-\frac{9}{2}\cos 2\theta\right)d\theta=\left[\frac{41}{2}\theta-\frac{9}{4}\sin 2\theta\right]_0^{\pi/2}=\left(\frac{41\pi}{4}-0\right)-(0-0)=\frac{41\pi}{4}$$

10. $A=\int_0^{2\pi}\frac{1}{2}r^2\,d\theta=\int_0^{2\pi}\frac{1}{2}[3(1+\cos\theta)]^2 d\theta$

$$=\frac{9}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=\frac{9}{2}\int_0^{2\pi}\left[1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$=\frac{9}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}=\frac{27}{2}\pi$$

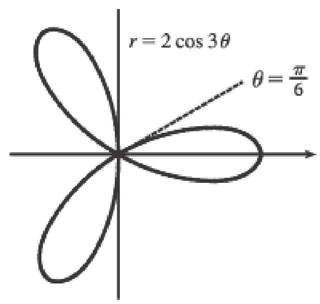


13. One-sixth of the area lies above the polar axis and is bounded by the curve $r=2\cos 3\theta$ for $\theta=0$ to $\theta=\pi/6$.

$$A=6\int_0^{\pi/6}\frac{1}{2}(2\cos 3\theta)^2\,d\theta=12\int_0^{\pi/6}\cos^2 3\theta\,d\theta$$

$$=\frac{12}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta$$

$$=6\left[\theta+\frac{1}{6}\sin 6\theta\right]_0^{\pi/6}=6\left(\frac{\pi}{6}\right)=\pi$$



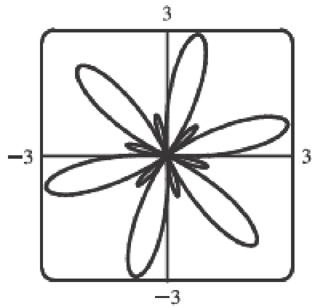
15. $A=\int_0^{2\pi}\frac{1}{2}(1+2\sin 6\theta)^2\,d\theta=\frac{1}{2}\int_0^{2\pi}(1+4\sin 6\theta+4\sin^2 6\theta)\,d\theta$

$$=\frac{1}{2}\int_0^{2\pi}\left[1+4\sin 6\theta+4\cdot\frac{1}{2}(1-\cos 12\theta)\right]d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}(3+4\sin 6\theta-2\cos 12\theta)\,d\theta$$

$$=\frac{1}{2}\left[3\theta-\frac{2}{3}\cos 6\theta-\frac{1}{6}\sin 12\theta\right]_0^{2\pi}$$

$$=\frac{1}{2}\left[\left(6\pi-\frac{2}{3}-0\right)-\left(0-\frac{2}{3}-0\right)\right]=3\pi$$



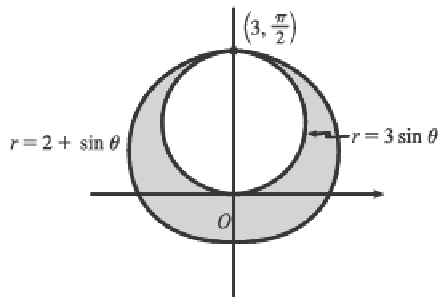
20. $r=0\ \Rightarrow\ 2\sin 6\theta=0\ \Rightarrow\ 6\theta=0\text{ or }\pi\ \Rightarrow\ \theta=0\text{ or }\frac{\pi}{6}.$

$$A=\int_0^{\pi/6}\frac{1}{2}(2\sin 6\theta)^2\,d\theta=\int_0^{\pi/6}2\sin^2 6\theta\,d\theta=2\int_0^{\pi/6}\frac{1}{2}(1-\cos 12\theta)\,d\theta=\left[\theta-\frac{1}{12}\sin 12\theta\right]_0^{\pi/6}=\frac{\pi}{6}$$

26. To find the shaded area A, we'll find the area A_1 inside the curve $r = 2 + \sin\theta$ and subtract $\pi\left(\frac{3}{2}\right)^2$ since $r = 3\sin\theta$ is a circle with radius $\frac{3}{2}$.

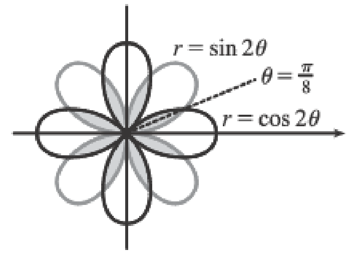
$$\begin{aligned} A_1 &= \int_0^{2\pi} \tfrac{1}{2}(2+\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi}(4 + 4\sin\theta + \sin^2\theta)\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left[4 + 4\sin\theta + \tfrac{1}{2}\cdot(1-\cos 2\theta)\right]d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left(\tfrac{9}{2} + 4\sin\theta - \tfrac{1}{2}\cos 2\theta\right)d\theta \\ &= \tfrac{1}{2}\left[\tfrac{9}{2}\theta - 4\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{1}{2}[(9\pi - 4) - (-4)] = \tfrac{9\pi}{2} \end{aligned}$$

So $A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$.



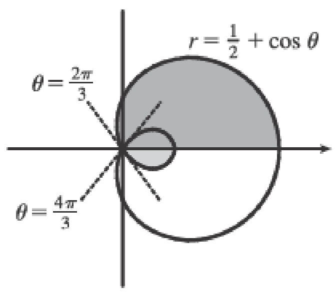
31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow$
$\theta = \frac{\pi}{8} \quad\Rightarrow$

$$\begin{aligned} A &= 8\cdot 2\int_0^{\pi/8}\tfrac{1}{2}\sin^2 2\theta\,d\theta = 8\int_0^{\pi/8}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta \\ &= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{\pi}{2} - 1 \end{aligned}$$



35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.

$$\begin{aligned} A &= 2\left[\int_0^{2\pi/3}\tfrac{1}{2}\left(\tfrac{1}{2}+\cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi}\tfrac{1}{2}\left(\tfrac{1}{2}+\cos\theta\right)^2 d\theta\right] \\ &= \int_0^{2\pi/3}\left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right)d\theta - \int_{2\pi/3}^{\pi}\left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right)d\theta \\ &= \int_0^{2\pi/3}\left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\ &\qquad - \int_{2\pi/3}^{\pi}\left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\ &= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi} \\ &= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) \\ &= \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right) \end{aligned}$$



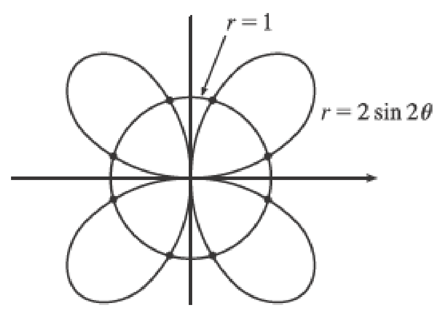
39. $2\sin 2\theta = 1 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

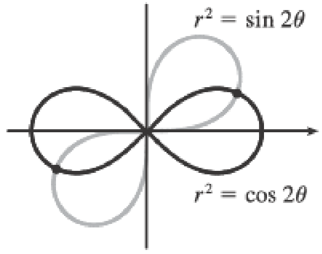
$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

[There are many ways to describe these points.]



42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \;\Rightarrow$ $\tan 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations] $\Rightarrow\; \theta = \frac{\pi}{8} + n\pi \;\Rightarrow\; \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



45. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi/3} \sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = \int_0^{\pi/3} \sqrt{9(\sin^2\theta + \cos^2\theta)}\, d\theta$

$= 3\int_0^{\pi/3} d\theta = 3\big[\,\theta\,\big]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) = \pi.$

As a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right) = 3\pi$, and since $\theta = 0$ to $\pi = \frac{\pi}{3}$ traces out $\frac{1}{3}$ of the circle (from $\theta = 0$ to $\theta = \pi$), $\frac{1}{3}(3\pi) = \pi$.

52. The curve $r = 1 + \cos\left(\frac{\theta}{3}\right)$ is completely traced with $0 \le \theta \le 6\pi$.

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[1 + \cos\left(\frac{\theta}{3}\right)\right]^2 + \left[-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\right]^2 \;\Rightarrow$

$L = \int_0^{6\pi} \sqrt{\left[1 + \cos\left(\frac{\theta}{3}\right)\right]^2 + \frac{1}{9}\sin^2\left(\frac{\theta}{3}\right)}\, d\theta$

≈ 19.6676