#5 Taylor's Series for $\cos x$ @ $a=\frac{\pi}{2}$

$f(x) = \cos x \qquad f(\frac{\pi}{2}) = 0$

$f^{(1)}(x) = -\sin x \qquad f^{(1)}(\frac{\pi}{2}) = -1$

$f^{(2)}(x) = -\cos x \qquad f^{(2)}(\frac{\pi}{2}) = 0$

$f^{(3)}(x) = \sin x \qquad f^{(3)}(\frac{\pi}{2}) = 1$

$\vdots$ Repeat

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

$$= \underbrace{0}_{n=0} + \underbrace{(-1)(x-\tfrac{\pi}{2})^1}_{n=1,\ k=0} + \underbrace{0}_{n=2} + \underbrace{\frac{(1)(x-\frac{\pi}{2})^3}{3!}}_{n=3,\ k=1} + \underbrace{0}_{n=4} + \underbrace{\frac{(-1)(x-\frac{\pi}{2})^5}{5!}}_{n=5,\ k=2} + \cdots + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\left(x-\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!} = -\left(x-\frac{\pi}{2}\right) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \cdots + \frac{(-1)^{k+1}(x-\frac{\pi}{2})^{2k+1}}{(2k+1)!} + \cdots$$

#23 Use #5 to estimate $\cos 80^\circ$ to 5 decimal places.

$$80^\circ = (80^\circ)\left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{4\pi}{9}$$

Scratch: $\frac{4\pi}{9} - \frac{\pi}{2} = \frac{8\pi - 9\pi}{18} = -\frac{\pi}{18}$

$$\therefore \cos 80^\circ \approx -\left(-\frac{\pi}{18}\right) + \frac{(-\frac{\pi}{18})^3}{3!} - \frac{(-\frac{\pi}{18})^5}{5!} + \frac{(-\frac{\pi}{18})^7}{7!}$$

$$f := n \to \frac{(-1)^{n+1}\cdot\left(-\frac{\mathrm{Pi}}{18}\right)^{2\cdot n+1}}{(2\cdot n+1)!}$$

$$n \to \frac{(-1)^{n+1}\left(-\frac{1}{18}\pi\right)^{2n+1}}{(2n+1)!}$$

$evalf(f(0))$

0.1745329252

$evalf(f(1))$

−0.0008860961561

$evalf(f(2))$

0.000001349601624

$\sum_{k=0}^{2} f(k)$

$$\frac{1}{18}\pi - \frac{1}{34992}\pi^3 + \frac{1}{226748160}\pi^5$$

$evalf(\%)$

0.1736481786

$\cos\left(\frac{4\cdot \mathrm{Pi}}{9}\right)$

$$\cos\left(\frac{4}{9}\pi\right)$$

$evalf(\%)$

0.1736481773

Maple says The 1st 2 terms ought to be plenty, since the 3rd term ($(x-\frac{\pi}{2})^5$-term) is smaller than the allowable error.

Too far. ~~n=1~~ n=1 gives k=0, k=1, i.e. 2 terms.

Fix on next page

$$\sum_{k=0}^{1} f(k)$$

$$\frac{1}{18}\pi - \frac{1}{34992}\pi^3$$

$evalf(\%)$

$$0.1736468290$$

$$\cos\left(\frac{4 \cdot \mathrm{Pi}}{9}\right)$$

$$\cos\left(\frac{4}{9}\pi\right)$$

$evalf(\%)$

$$0.1736481773$$

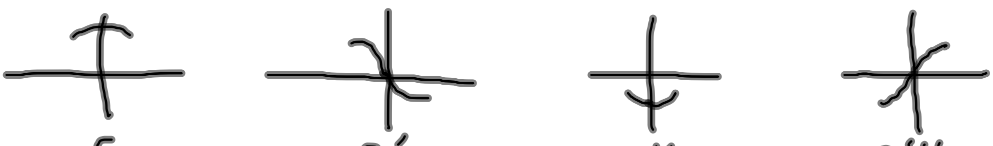
We used Alternating Series Test for this.
what about Taylor's Inequality?

$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!}$, where M is a bound on $|f^{(n+1)}(x)|$ on the interval in question, in this case

$|R_1(x)| \le \frac{M\,|x-a|^2}{2!}$ $a = \frac{\pi}{2}$, x in question is $\frac{4\pi}{9}$

$d = |x-a| = \left|\frac{4\pi}{9} - \frac{\pi}{2}\right| = \frac{\pi}{18}$

The interval in question for testing $\left|\frac{d^n}{dx^n}[\cos x]\right|$ would be $\left[-\frac{\pi}{18}, \frac{\pi}{18}\right]$



f	f'	f''	f'''
max @ $x=0$	max @ endpts	max @ $x=0$	max @ endpts.
$\cos(0)=1$	$\sin(\frac{\pi}{18})$	$-\cos(0)$	$\sin(\frac{\pi}{18})$
1	1	1	1

$M=1$ is fine. You always overthink, Steve.

$$|R_1(x)| \le \frac{1\,|x-2|^2}{2!} = \frac{\left(\frac{\pi}{18}\right)^2}{2!}$$

$$|R_2(x)| \le \frac{1\left(\frac{\pi}{18}\right)^3}{3!}$$

This is harder than Alternating Series Test.