12.10 #26

$$f(x) = \frac{1}{(1+x)^4} = (1+x)^{-4}.$$

Expand as power series. Find Radius of Convergence.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 - 4x + \frac{-4(-5)}{2!}x^2 + \frac{-4(-5)(-6)}{3!}x^3 + \cdots$$

$$= 1 - 4x + \frac{5\cdot4}{2!}x^2 - \frac{6\cdot5\cdot4}{3!}x^3 + \frac{7\cdot6\cdot5\cdot4}{4!}x^4 - \frac{8\cdot7\cdot6\cdot5\cdot4}{5!}x^5$$

$$\binom{6}{3} = \frac{6!}{3!\,3!} \qquad \binom{7}{4} \to \binom{7}{3} \qquad \binom{8}{5} = \frac{8!}{3!\,5!} \to \binom{8}{3}$$

$$= \sum_{k=0}^{\infty} \binom{k+3}{3} (-1)^k x^k$$

Joel

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\binom{n+1+3}{3}}{\binom{n+3}{3}} \frac{x^{n+1}}{x^n} \right|$$

$$\left| \frac{\frac{(n+4)!}{(n+1)!\,3!}}{\frac{(n+3)!}{n!\,3!}} x \right| = \frac{n+4}{n+1}|x| \xrightarrow{n\to\infty} |x|$$

Need $|x| < 1 \equiv R$

[17] **THE BINOMIAL SERIES** If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Book Ans:

$$\sum_{k=0}^{\infty} \binom{-4}{k} x^k$$

$$\binom{-4}{n} = \frac{-4(-5)(-6)\cdots(-4-n+1)}{n!} =$$

$$= \frac{-4(-5)(-6)\cdots(-(n+3))}{n!}$$

$$= (-1)^n \frac{(n+3)(n+2)(n+1)}{3!} = (-1)^n \binom{n+3}{3} = (-1)^n \binom{n+3}{n}$$

1
1 1
1 2 1
1 3 3 1

$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$k + (n-k) = n$$

$$C(10,3) = \binom{10}{3} = \frac{10!}{7!\,3!} \qquad \binom{10}{7} = \frac{10!}{3!\,7!}$$

$\shortparallel$

${}_{10}C_3$

$\shortparallel$

C^{10}_{3}

12.11 APPLICATIONS OF TAYLOR POLYNOMIALS

If f has a Taylor's series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The n^{th} degree Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

And T_1 is simply the ***tangent line to f at $x = a$*** !!!

$$T_1(x) = f(a) + f'(a)(x-a)$$

$$= m_{tan}(x-a) + f(a)$$

We've seen in Calculus I how useful tangent line approximation can be in a small neighborhood of $x = a$. Basically, what we're saying *now* is that we can do the same thing with more accuracy by taking the quadratic, cubic, quartic, quintic, ... Taylor polynomial to f at $x = a$. The higher the degree, generally, the greater the accuracy over a wider radius (centered at $x = a$.

How good is the approximation? We look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

Taylor's Inequality
Alternating Series Test

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases we can use Taylor's Inequality (Theorem 12.10.9), which says that if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

There are at least two versions of these "type questions:"

1. How big must n be in order to approximate f to within the desired tolerance on a particular interval?
2. Given n is fixed, how close to a must x be in order to be within the desired tolerance?

One relating to the first question is #25:

25. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

One relating to the second question is #26:

26. How many terms of the Maclaurin series for $\ln(1+x)$ do you need to use to estimate ln 1.4 to within 0.001?

If I recall correctly, you can get away with an Alternating Series approach to # 26, since it's cousin to an alternating geometric series.

I'm assigning #1. I will work a very similar problem, using Maple.

1. (a) Find the Taylor polynomials up to degree 6 for $f(x) = \cos x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
(b) Evaluate f and these polynomials at $x = \pi/4$, $\pi/2$, and π.
(c) Comment on how the Taylor polynomials converge to $f(x)$.

Find the Taylor polynomials up to degree 6 for

$f(x) = \sin x$

centered at $a = 0$. Graph f and these polynomials on a common screen.

(b) Evaluate f and these polynomials at $x = \pi/4$, $\pi/2$, and π.

(c) Comment on how the Taylor polynomials converge to $f(x)$.

Let me e-mail you the assignments
for 12.11 Sorry...

3–10 Find the Taylor polynomial $T_n(x)$ for the function f at the number a. Graph f and T_3 on the same screen.

4. $f(x) = x + e^{-x}, \quad a = 0$

CAS **11–12** Use a computer algebra system to find the Taylor polynomials T_n centered at a for $n = 2, 3, 4, 5$. Then graph these polynomials and f on the same screen.

12. $f(x) = \sqrt[3]{1 + x^2}, \quad a = 0$

13–22

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(c) Check your result in part (b) by graphing $|R_n(x)|$.

18. $f(x) = \ln(1 + 2x), \quad a = 1, \quad n = 3, \quad 0.5 \leqslant x \leqslant 1.5$

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1}$$