An example that bootstraps off the binomial expansion, rather than just using the basic algorithm is given by:

V EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Now just apply the binomial expansion to the above. I never would've thought of this as a student.

17 THE BINOMIAL SERIES If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$+ \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots$$

A table of the standard Maclaurin series. Probably good material for a cheat sheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

(6) $f(x) = \ln(1+x)$ Did in 12.9 by bootstrapping off of $\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$ Maclaurin's (centered @ $x=0$)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$ Taylor's expansion centered @ $x=a$

$f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{x+1} = (x+1)^{-1}, \quad f'(0) = 1$ $\qquad 0! = 1$

$f''(x) = -(x+1)^{-2}, \quad f^{(2)}(0) = -1$

$f^{(3)}(x) = 2(x+1)^{-3}, \quad f^{(3)}(0) = 2 = 2!$

$f^{(4)}(x) = -3 \cdot 2\,(x-1)^{-4}, \quad f^{(4)}(0) = -3!$

$\vdots$

$f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + 1x - 1\frac{x^2}{2!} + \frac{2!\,x^3}{3!} - \frac{3!\,x^4}{4!} + \cdots,$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(n-1)!\,x^n}{n!} = \boxed{\sum_{n=0} (-1)^{n+1} \frac{x^n}{n}} =$$ House Keeping.

$n=0$!?

12.9

$\ln(1+x) = f(x)$

$$\frac{d}{dx}\left[\ln(1+x)\right] = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=1}^{\infty} (-x)^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = f'(x).$$ Want $f(x)$. Integrate:

$$f(x) = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} =$$

House Keeping

Handle $n=0$

(17) $f(x) = \cos x$ centered @ $x = \pi$

$f(x) = \cos x$ $\quad \cos \pi = -1$

$f'(x) = -\sin x$ $\quad -\sin \pi = 0$

$f''(x) = -\cos x$ $\quad -\cos \pi = 1$

$f'''(x) = \sin x$ $\quad \sin \pi = 0$

$$\sum_{n=0}^{\infty} f^{(n)}(\pi) \frac{(x-\pi)^n}{n!}$$

$$= \underset{k=0}{-1 \frac{(x-\pi)^0}{0!}} + \cancel{0 \frac{(x-\pi)^1}{1!}} + \underset{k=1}{1 \frac{(x-\pi)^2}{2!}} + \cancel{0} \underset{k=2}{-1 \frac{(x-\pi)^4}{4!}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (x-\pi)^{2k}}{(2k)!} \quad . \quad R = \infty$$

(Alternating Series) $\quad \boxed{\frac{x^n}{n!} \xrightarrow{n \to \infty} 0 \quad \forall x \in \mathbb{R}}$

That's fine for $x > \pi$, but for $x < \pi$? Still alternating b/c of $(x-\pi)^{2k} \geq 0$ always, so the $(-1)^{n+1}$ assures alternating sign.

TABLE 1 we can bootstrap off these standard Maclaurin's Series, as in §12.9, with $\frac{1}{1-x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

So $$e^{-x^2} + \cos x = \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n}}{n!} + \frac{(-1)^n x^{2n}}{(2n)!} \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n!} + \frac{1}{(2n)!} \right) x^{2n}$$

Book writes out the first few terms. This is what you'd do to approximate $e^{-x^2} + \cos x$ at a particular value with only an abacus.

See #43 Only need a few terms for approximating the integral

#43

$e^{-.2} \approx \sum_{n=0}^{M?} \frac{(-.2)^n}{n!} = 1 + (-.2) + \frac{(-.2)^2}{2} + \frac{(-.2)^3}{3!}$

To 5 decimal places.

$= 1 - .2 + \frac{.04}{2} - \frac{.008}{6} + \frac{.0016}{24} - \frac{(.2)^5}{5!} + \frac{(-.2)^6}{6!}$

6.7×10^{-5} $8.\overline{8} \times 10^{-8}$

$(2 \times .1)^4 = 2^4 \times .1^4 = 2^4 \times 10^{-4}$

16×10^{-4}

Far enough.

So $e^{-.2} \approx .81873$ 0 0.0016

There are some more challenging exercise types we didn't cover today:

Term-by-Term Long Division of Series.