12.10 TAYLOR AND MACLAURIN SERIES

Assume that f can be represented by a power series on some interval centered at $x = a$. Then

$$\boxed{1} \quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R$$

Where R is the radius of the interval in question. Note the interval described is an open interval. Typically, we'll need to check if the endpoints are included or not.

§12.10 #s 4, 5, 10, 11, 13, 18, 22, 26, 29, 34, 35
I
II 39, 44, 47, 51, 54, 55, 56, 63, 66

Here's the idea:

If f has such a representation, then $f(a)$ is given by

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \ldots = c_0$$

$$f(a) = c_0$$

Likewise, term by term differentiation reveals that

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R$$

$$f'(a) = c_1$$

$$\boxed{3} \quad f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2$$

$$\frac{f''(a)}{2} = \frac{f^{(2)}(a)}{2} = c_2$$

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$$

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

$$\frac{f'''(a)}{3 \cdot 2} = \frac{f^{(3)}(a)}{3!} = c_3$$

In General, then:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This gives us a nice algorithm for calculating the coefficients of a power series for *any* function that we can differentiate! COOL!

$\boxed{5}$ **THEOREM** If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Rewriting the power series expansion for f using this gives us:

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

When the series is centered at $a = 0$, it's called the Maclaurin series or Maclaurin expansion for f. It's generally more convenient to use $a = 0$, especially when the radius of convergence is infinite.

Note for Future: But typically, we want to choose a fairly close to values of x at which we are likely to be evaluating f, because the closer x is to a, typically the faster the convergence of the series and the fewer terms we need in order to approximate f by a partial sum.

Maclaurin series

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

The industry standard Maclaurin series example

EXAMPLE 1 Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

$f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1, \ldots\ldots$

$\boxed{0! = 1}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

what's the radius of convergence?

$R = \infty$, $I = \mathbb{R}$

Pretty neat stuff. Just keep in mind, we've been *assuming* that f actually *has* a power series representation.

As our first requirement for the existence of such a representation, for this method to work, **clearly f must have derivatives of all orders**. (necessary, but not sufficient).

T_n is the n^{th} degree Taylor Polynomial. You'd like to think that as n grows large, that T_n gives a decent approximation for f. Follow the link to TEC, below for an illustration:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

In general, this representation "works" if T_n converges to f. So to see if such a thing is happening, we basically have to show that the remainder, R_n converges to zero in the following:

$$f(x) = \lim_{n\to\infty} T_n(x)$$

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

$\boxed{8}$ **THEOREM** If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

A tool for showing that R_n converges to zero (or for deciding how large n must be in order for T_n to be sufficiently close to f...

$\boxed{9}$ **TAYLOR'S INEQUALITY** If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leq d$$

So finding bounds on derivatives appears to be on the table.

A useful fact that we no longer have to justify every time we use it is the following:

$\boxed{10}$

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x}$$

EXAMPLE 4 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

The Maclaurin series is nice for $\sin x$, because the pattern is relatively easy to see and half the terms just vanish on their own. The Taylor series expansion for $\sin x$ doesn't generally have these terms vanish. The text does a nice writeup for one of these (and for cosine, too).

In fact, some of the examples the book worked out for you would've been nice exercises. It's almost too bad.

$f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f^{(3)}(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin x$

Repeats.

$$0 + \frac{1}{1}x^1 + 0 - \frac{1}{3!}x^3 + 0 - \frac{1}{5!}x^5 + \cdots$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots +$$

$k=0 \quad 1 \quad 2 \quad 3 \quad k=n$

$$(-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$(-1)^0 \frac{x^{2(0)+1}}{(2(0)+1)!} \; ? \qquad \frac{x^1}{1} = x \; \checkmark$$

The Binomial Coefficient, also known as combinations of n things taken k at a time, is generalized quite a bit in this section, for those of you who've already been exposed to combinations, permutations, and the Binomial Theorem in previous courses.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Most Beginning Stats.

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7.6.5}{3!} = \frac{7\cdot 6 \cdots (7-3+1)}{3!}$$

$$\binom{7}{2} = \frac{7!}{2!5!} = \frac{7.6}{2!} = \frac{7 \cdots (7-2+1)}{2!}$$

$$\binom{7}{5} = \frac{7!}{5!2!} = \frac{7.6}{2!} = \frac{7\cdot 6 \cdots (7-5+1)}{5!}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

This generalizes better for things like:

$$\binom{\sqrt{7}}{11} = \frac{\sqrt{7}(\sqrt{7}-1)(\sqrt{7}-2)\cdots(\sqrt{7}-10)}{11!}$$

Too Weird!

This is discussed in Example 8.

17 THE BINOMIAL SERIES If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$(k-k)$

$x^{k(k-1)}$

0

What happens when k is a positive integer?

$$(x+2)^5 = \sum_{n=0}^{5} \binom{5}{n} x^n = 1x^5 + 5x^4 \cdot 2 + 10x^3 \cdot 2^2 + 10x^2 \cdot 2^3 + 5x \cdot 2^4 + 2^5$$

By Binomial Theorem in MAT 121.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$

An example that bootstraps off the binomial expansion, rather than just using the basic algorithm is given by:

EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

f, f', f'' . . . Not necessary

Now just apply the binomial expansion to the above. I never would've thought of this as a student.

A table of the standard Maclaurin series. Probably good material for a cheat sheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$