DIFFERENTIATION AND INTEGRATION OF POWER SERIES

The upshot of all this discussion is...

You may safely differentiate and integrate a convergent power series term-by-term, just as you might hope and expect you could!

2 THEOREM If the power series $\Sigma\, c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

This has applications peeking around corners left and right. You'd be surprised.

Pattern recognition is going to be HUGE, here. But first, some notational tweaks that might be more along the lines of some practical applications:

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n]$

(iv) $\displaystyle\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$

More Spinoffs:

23–26 Evaluate the indefinite integral as a power series. What is the radius of convergence?

24. $\displaystyle\int \frac{\ln(1-t)}{t}\,dt$

$$\frac{d}{dt}\left[\ln(1-t)\right] = -\frac{1}{1-t}$$

So, $\displaystyle \ln(1-t) = \int -\frac{1}{1-t}\,dt + C$ 0

$$= -\int (1 + t + t^2 + t^3 + \cdots)\,dt$$

$$= -\int \left(\sum_{n=0}^{\infty} t^n\right) dt = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} + C \text{ is } 0$$

$$\therefore \quad \frac{\ln(1-t)}{t} = \frac{1}{t}\left(-\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\right)$$

$$= -\sum_{n=0}^{\infty} \frac{t^n}{n+1}$$

$$\therefore \int \frac{\ln(1-t)}{t}\,dt = -\int \sum_{n=0}^{\infty} \frac{t^n}{n+1}\,dt =$$

$$= -\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2} + \hat{C}$$

27–30 Use a power series to approximate the definite integral to six decimal places.

See also: Example 8.

27. $\int_0^{0.2} \frac{1}{1+x^5}\,dx$

$.000005$

to nearest 100^{th}:
Multiply by 100.
Add .5
Truncate the decimal part
Divide by 100

90.267
9026.7
9027.2
9027
90.27

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n$$

$$\int_0^{.2} \frac{1}{1+x^5}\,dx = \int_0^{.2} \left(\sum_{n=0}^{\infty} (-1)^n x^{5n} \right) dx$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1} \right]_0^{.2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{.2^{5n+1}}{5n+1} = \underset{n=0}{\frac{.2}{1}} - \underset{n=1}{\frac{.2^6}{6}} + \underset{n=2}{\frac{.2^{11}}{11}} + \cdots$$

$$= .2$$

$.000005 = 5 \times 10^{-6}$ is ceiling on error
Find term less than this in absolute value.

NOTE $\frac{.2^{11}}{11} = 1.86\overline{18} \times 10^{-9} < 5 \times 10^{-6}$ ✓
$\cancel{3}$ $\to b_2$

So, $S_1 = .2 - \frac{.2^6}{6} \approx .199989$ does it.

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(a) $\sum_{n=1}^{\infty} nx^{n-1}$ $|x|<1$ Find the sum.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so the above is $\frac{d}{dx}\left[\frac{1}{1-x}\right] = -\frac{1}{(1-x)^2}$

Joel

(b) Find the sum:

(i) $\sum_{n=1}^{\infty} nx^n$, $|x|<1$ Looks like $x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$!

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ $\sum_{n=1}^{\infty} n\cdot\frac{1}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$

(c) Find the sum:

(i) $\sum_{n=2}^{\infty} n(n-1)x^n$, $|x|<1$

is the 2nd derivative of $\frac{1}{1-x}$ times x^2 (after the fact)

$f(x) = x^n$

$f'(x) = nx^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right]$

$\frac{x^2}{(1-x)^3}$

$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \sum_{n=0}^{\infty} nx^{n-1}$

$\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] = \sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$

Mills can't differentiate $(1-x)^2$

So $\sum_{n=2}^{\infty} n(n-1)x^n$ = previous times x^2

$= \frac{1}{(1-x)^3}\cdot x^2 = \frac{x^2}{(1-x)^3}$

So $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$

$= \sum_{n=2}^{\infty} n(n-1)\cdot\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3}$

Good Job Class for catching that.