What does $\frac{1}{1-r}$ remind you of?

$\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$ for $|r| < 1$

$\frac{1-r^n}{1-r} \xrightarrow{n \to \infty} \frac{1}{1-r}$

$|r| < 1$ needed.

How many wrinkles can we GET outta this?

12.9 REPRESENTATIONS OF FUNCTIONS AS POWER SERIES

Here we use some of the things we know from before about geometric series, in particular, to help us write all KINDS of functions as power series. We start with our old friend, the closed-form expression for the geometric series. Recall:

$$a + ar + ar^2 + ar^3 + \cdots = a\sum_{n=0}^{\infty} r^n = a\sum_{n=1}^{\infty} r^{n-1} = a \cdot \frac{1}{1-r}$$

In particular, $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

Turn it around, and *now* we look for a power series representation for objects like

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Now, there's NOTHING that says we can NOT simply do a change of variable, if we like, and every reason to hope that we CAN!

Consider $\dfrac{1}{1-x^2}$

If life were good and fair, we *would* be able to simply replace *x* by x^2 in the preceding... WE CAN!!!!

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots$$

$$= 1 + x^2 + x^4 + x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} x^{2n} \qquad !!!!!!$$

Example 1 does something very similar with

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

3–10 Find a power series representation for the function and determine the interval of convergence.

5. $f(x) = \frac{2}{3-x} = \frac{2}{3(1-\frac{x}{3})} = \frac{2}{3} \cdot \frac{1}{1-\frac{x}{3}}$

$= \frac{2}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots + \left(\frac{x}{3}\right)^n + \cdots\right]$

$= \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \frac{2}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n}$

↳ Converges for $\left|\frac{x}{3}\right| < 1$

$|x| < 3 = R$

$I = (-3, 3)$

8. $f(x) = \dfrac{x}{2x^2+1} = \dfrac{x}{1+2x^2} = \dfrac{x}{1-(-2x^2)}$

$= x\left[\dfrac{1}{1-(-2x^2)}\right] = x\sum_{n=0}^{\infty}(-2x^2)^n$

$= x\left[\sum_{n=0}^{\infty}(-1)^n 2^n (x^2)^n\right]$

$= x\sum_{n=0}^{\infty}(-1)^n \cdot 2^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$

Brent

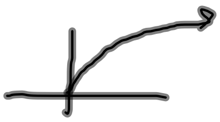
$= x - 2x^3 + 2^2x^5 - 2^3x^7 + \cdots + (-1)^n \cdot 2^n x^{2n+1}$

$\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{2^{n+1}x^{2n+3}}{2^n x^{2n+1}}\right| = 2x^2 \overset{\text{want}}{<} 1$

$x^2 < \frac{1}{2}$

$\rightarrow \sqrt{x^2} < \sqrt{\frac{1}{2}} \Rightarrow |x| < \frac{\sqrt{2}}{2} = R$

for absolute convergence



$I = ?$

$\frac{\sqrt{2}}{2} - 2\left(\frac{\sqrt{2}}{2}\right)^3 + 2^2\left(\frac{\sqrt{2}}{2}\right)^5 - \cdots$

$= \frac{\sqrt{2}}{2} - \frac{2}{2^3}\cdot 2\sqrt{2} + \frac{2^2}{2^5}\cdot 2^2\sqrt{2} - 2^3\cdot\left(\frac{\sqrt{2}}{2}\right)^7$

Joel

$= \frac{\sqrt{2}}{2} - \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} - \frac{2^3 \cdot 2^3\sqrt{2}}{2^7} + \cdots$

Nope

$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \cdots$

Doesn't converge at $x = \frac{\sqrt{2}}{2}$ or $x = -\frac{\sqrt{2}}{2}$

$$I = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

11–12 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

12. $f(x) = \dfrac{x+2}{2x^2 - x - 1} = \dfrac{1}{x-1} - \dfrac{1}{2x+1}$

$= -\dfrac{1}{1-x} - \dfrac{1}{1-(-2x)}$

$= -\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (-2x)^n \to \sum (-2)^n x^n$

$= -\left[1 + x + x^2 + \cdots + 1 - 2x + 2^2x^2 - 2^3x^3 + \cdots\right]$

Not too easy to proceed

$= -\sum_{n=0}^{\infty} (1 + (-2)^n) x^n$ Adding term-by-term.

$-\sum_{n=0}^{\infty} x^n$ has $r_1 = 1$, $I = (-1, 1)$

$-\sum_{n=0}^{\infty} (-2)^n x^n = -\sum_{n=0}^{\infty} (-1)^n 2^n x^n$

Geometric Series! Need $|2x| < 1$

$|x| < \frac{1}{2}$

$R = \frac{1}{2}$ for combined series.

$I = (-\frac{1}{2}, \frac{1}{2})$

DIFFERENTIATION AND INTEGRATION OF POWER SERIES

The upshot of all this discussion is...

You may safely differentiate and integrate a convergent power series term-by-term, just as you might hope and expect you could!

2 THEOREM If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

This has applications peeking around corners left and right. You'd be surprised.

Pattern recognition is going to be HUGE, here. But first, some notational tweaks that might be more along the lines of some practical applications:

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$

(iv) $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$

There are a few interesting surprises these results give us. We can really expand our flight envelope.

15–18 Find a power series representation for the function and determine the radius of convergence.

15. $f(x) = \ln(5 - x)$

$$f'(x) = -\frac{1}{5-x} = -\frac{1}{5}\cdot\frac{1}{1-\left(\frac{x}{5}\right)}$$

$$= 1 + \frac{x}{5} + \left(\frac{x}{5}\right)^2 + \cdots + \frac{x^n}{5^n} + \cdots$$

So $f(x) =$

$$\int\left(1 + \frac{x}{5} + \frac{x^2}{5^2} + \cdots + \frac{x^n}{5^n} + \cdots\right)dx$$

$$= C + x + \frac{x^2}{2\cdot 5} + \frac{x^3}{3\cdot 5^2} + \cdots + \frac{x^{n+1}}{5^n(n+1)} + \cdots$$

$$= \int\sum_{n=0}^{\infty}\frac{x^n}{5^n}\,dx = \sum\int\frac{x^n}{5^n}\,dx = \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} + C$$

You gotta be kiddin' me!!! Ahhhh. We *do* know the derivative of this function! And its derivative has a nice power series representation. So to find the representation for the function, we take the antiderivative, term-by-term of the series that we already know! Sweet!

And think of the possibilities for estimating integrals of some uglier functions that have warm 'n' fuzzy power series representations... You just integrate the power series term by term (power rule every term), and use as many terms as you need.

$$\ln(5-x)\Big|_{x=0} = C = \ln 5$$

$$\therefore \quad \boxed{\ln(5-x) = \ln 5 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^n}}$$

12.9 #s 4, 7, 10, 11, 13, 14, 17, 18, 25, 29, 32, 35 → Recall Theorem 10.4.2

#s 19–22 COOL MAPLE

#s 38–40 Advanced Calc.

More Spinoffs:

23–26 Evaluate the indefinite integral as a power series. What is the radius of convergence?

24. $\int \frac{\ln(1 - t)}{t}\, dt$

27–30 Use a power series to approximate the definite integral to six decimal places.

See also: Example 8.

27. $\int_0^{0.2} \frac{1}{1+x^5}\,dx$