12.8 #s 1, 2, 4, 7, 10, 13, 16, 19, 22, 25, 28, 32, 33
#s 38 - 42: For presentation in class:
Write up your problem. You will use your write-up and the document camera to discuss it. Bring it in on Monday. I'll check it out and if it's OK, you may present on Tuesday. Counts as an extra homework.

12.8 Power Series

Each term is a power function.

$\boxed{1}$ $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

A power series whose coefficients are the same constant is just a geometric series, which converges for $-1 < x < 1$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\sum_{n=0}^{\infty} cx^n = c + cx + cx^2 + cx^3 + \cdots = c\sum_{n=0}^{\infty} x^n$$

A power series in $x - a$ or "centered at $x = a$". You're basically shifting what you know about a power series in x either right ($a > 0$) or left ($a < 0$), the same as you shift $f(x)$ to the right 5 by replacing x by $x - 5$.

$\boxed{2}$ $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Heretofore, we had terms in which the only variable was the n in each term. Now, when we talk about convergence, we will want to know the values of x for which the series converges

For what values of x is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent?

Ratio Test: Converges if $\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow{n\to\infty} c < 1$

$$\left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \xrightarrow{n\to\infty} 0 < 1$$

converges for $\underline{\text{any}}$ x (absolutely)

Radius of convergence: ∞

Interval of convergence (Centered at a)

$a=0$ for this one. ($(x-0)^n$ each term)

$(-\infty, \infty)$

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}\right|$$

Scratch:

$$\left(\frac{2^{2n}}{2^{2(n+1)}} = 2^{2n-2(n+1)} = 2^{2n-2n-2} = 2^{-2} \right)$$

$$= \left|\frac{x^2}{2^2(n+1)^2}\right| = \frac{1}{2^2(n+1)^2}|x| \xrightarrow{n\to\infty} 0 < 1$$

$\forall\, x \in \mathbb{R}$

Radius $= \infty$

Interval $= (-\infty, \infty)$

The language or parlance or lingo for dealing with power series is always in terms of convergence of partial sums, for instance,

$$J_0(x) = \lim_{n \to \infty} s_n(x)$$

Have a look at 12.8 Maple.

Same as $\sum_{n=1}^{\infty} x^{n-1}$

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	Degenerate $\{0\}$ Interval
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

12.8 EXERCISES

1. What is a power series?
2. (a) What is the radius of convergence of a power series? How do you find it?
   (b) What is the interval of convergence of a power series? How do you find it?

3–28 Find the radius of convergence and interval of convergence of the series.

$\boxed{3.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ 4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

$$\left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right|$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}} |x| \xrightarrow{n\to\infty} |x| \underset{<}{\text{NEED}} 1$$

So, $R = 1$

I: $|x| < 1$

$-1 < x < 1$

$I = (-1, 1)$

10. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

$$\left| \frac{10^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{10^n x^n} \right| = \frac{n^3}{(n+1)^3} \cdot |10x|$$

$$\xrightarrow{n \to \infty} |10x| \overset{\text{Need}}{<} 1 \implies$$

$$|x| < \frac{1}{10}$$

$$-\frac{1}{10} < x < \frac{1}{10}$$

$R = \frac{1}{10}$

$I = \left(-\frac{1}{10}, \frac{1}{10}\right)$

16. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{(x-3)^n}\right|$$

$$= \frac{2n+1}{2n+3} \cdot |x-3| \xrightarrow{n\to\infty} |x-3| < 1$$

$R = 1$

$$I: \quad -1 < x-3 < 1$$
$$+3 = +3 = +3$$
$$2 < x < 4$$

Converges absolutely for $x \in I = (2,4)$

NoT Quite

Check Endpoints:

$x = 2$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2-3)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

Comparison Test:

Convergent: Show it's term-by-term SMALLER than the terms of a convergent series.

Divergent: Show it's term by term BIGGER than the terms of a divergent series.

Can't use it. But I'm thinking, with "n^1" in denominator harmonic series.

Limit Comparison.

$$\frac{\frac{1}{2n+1}}{\frac{1}{n}} = \frac{n}{2n+1} \xrightarrow{n\to\infty} \frac{1}{2} \neq 0$$

Thx, Joel.

& so $\sum \frac{1}{2n+1}$ converges/diverges same as $\sum \frac{1}{n}$ does. $\sum \frac{1}{n}$ diverges. So does $\sum \frac{1}{2n+1}$ by LIMIT COMPARISON

16. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$

at $x=4$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(4-3)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

Converges Conditionally at $x=4$!

$R = 1$

$I = (2, 4]$

23. $\sum_{n=1}^{\infty} n!(2x-1)^n$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$