C11 Parametrics & Polars

Parametrics:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Polars:

$x = r\cos\theta \qquad y = r\sin\theta$

Give $r = f(\theta)$, we have

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[f(\theta)\sin\theta\right]$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[f(\theta)\cos\theta\right]$$

& $\frac{dy}{dx}$ as in parametrics!

§11.2 #11

$x = 4 + t^2,\ y = t^2 + t^3$

$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 3t^2 + 2t$

$$\therefore \frac{dy}{dx} = \frac{3t^2+2t}{2t} = \frac{3t+2}{2}$$

$$\frac{d^2y}{dt^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{3}{2t}$$

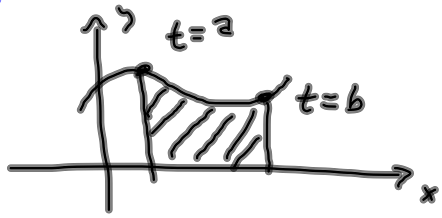
For what t is this curve concave up?

$$\frac{3}{2t} > 0 \iff t > 0$$

$$\{t : t > 0\}$$

Areas for parametrics

$\int_a^b y\,dx$ between curve & x-axis

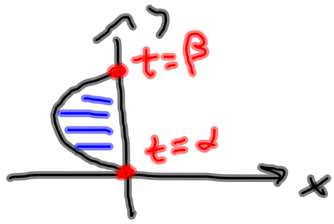


Given $x = f(t)$, $y = g(t)$

$dx = f'(t)\,dt$

Area $= \int_a^b g(t)f'(t)\,dt$

$\int_\alpha^\beta x\,dy = \int_\alpha^\beta f(t)g'(t)\,dt$



Example 11.2 II #32

Area between $x = t^2 - 2t$, $y = \sqrt{t}$ & y-axis.

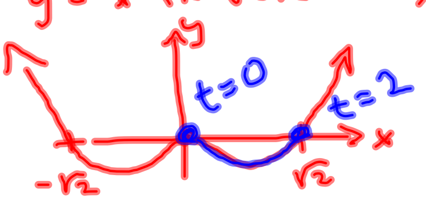
$x = t(t-2) = 0$ when $t = 0, 2$

$y = \sqrt{t} \Rightarrow y = 0, \sqrt{2}$

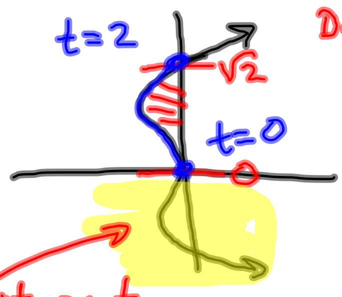
Eliminate parameter to "see" rectangular version:

$t = y^2 \Rightarrow x = y^2(y^2-2) = y^2(y-\sqrt{2})(y+\sqrt{2})$

$y = x^2(x-\sqrt{2})(x+\sqrt{2})$



$x = y^2(y-\sqrt{2})(y+\sqrt{2})$



Domain issue: $y = \sqrt{t}$, so $y < 0$ is stupid.

Stupit part.

$y = t^{\frac{1}{2}}$

$y' = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

$\int_0^2 x\,dy = \int_0^2 (t^2-2t)\left(\frac{1}{2\sqrt{t}}\right)dt$

$= \frac{1}{2}\int_0^2 \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$

Arc Length parametrics. Polars

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

41 $x = 1+3t^2$, $y = 4+2t^3$ $0 \le t \le 1$

$$\frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 6t^2$$

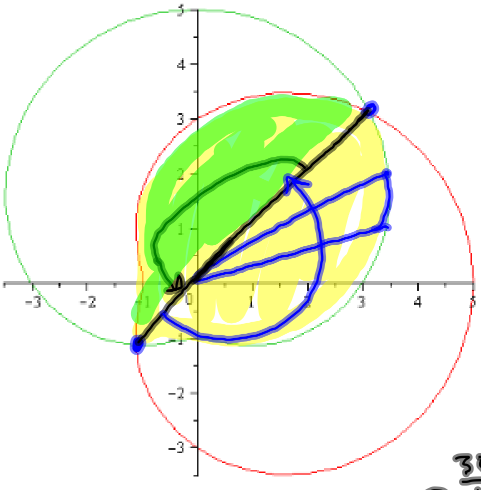
$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^1 6t\sqrt{1+t^2}\, dt = 3\int_0^1 \sqrt{1+t^2} \cdot 2t\,dt$$

$$3\int u^{\frac{1}{2}} du \ldots$$

$r = 3 + 2\cos\theta$, $r = 3+2\sin\theta$

Find area enclosed by the two.



$$3+2\cos\theta = 3+2\sin\theta$$
$$\cos\theta = \sin\theta$$
$$\theta = \frac{\pi}{4}, \ \theta = \frac{3\pi}{4}$$

$$\text{Area} = \frac{1}{2}\int_a^b [f(\theta)]^2 d\theta$$

$$\text{Green Area} = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (3+2\sin\theta)^2 d\theta$$

$$\text{Yellow Area} \quad \frac{1}{2}\int_0^{\frac{\pi}{4}} (3+2\cos\theta)^2 d\theta + \int_{\frac{3\pi}{4}}^{0} (3+2\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (3+2\cos\theta)^2 d\theta$$

Integral Test
b_{n+1} Alternating Series Test $\Bigg\}$ Bounds on error for $S_n \approx S$

Find N so that $a_n = \frac{\sqrt{n-1}}{n+5} < .01$ for any $n > N$.

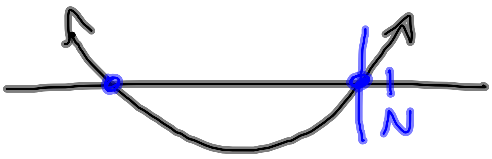
$$\frac{\sqrt{n-1}}{n+5} < .01$$

$$\sqrt{n-1} < .01(n+5)$$

$$n-1 < .0001(n^2+10n+25)$$

$$10000n - 10000 < n^2 + 10n + 25$$

$$n^2 - 9990n + 10025 > 0$$



Solve $n^2 - 9990n + 10025 = 0$
Pick N bigger than the right solution.

Bonus
Make $\frac{\sqrt{n-1}}{n+5} < \varepsilon$ for any ε

This Proves $\frac{\sqrt{n-1}}{n+5} \xrightarrow{n \to \infty} 0$

Make $\frac{n+5}{\sqrt{n-1}} > 100$

Find N such that $a_n > 100$ for every $n > N$.

$\boxed{a_n \xrightarrow{n \to \infty} \infty \text{ means ?}}$