12.5 ALTERNATING SERIES

12.5 #s 6, 7, 11-19, 22, 24, 29, 31-34

THE ALTERNATING SERIES TEST If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\boxed{\text{(i)} \quad b_{n+1} \leq b_n \qquad \text{for all } n}$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

or "eventually"
Maybe
$b_n \geq b_{n+1} \ \forall n \geq 100$

then the series is convergent.

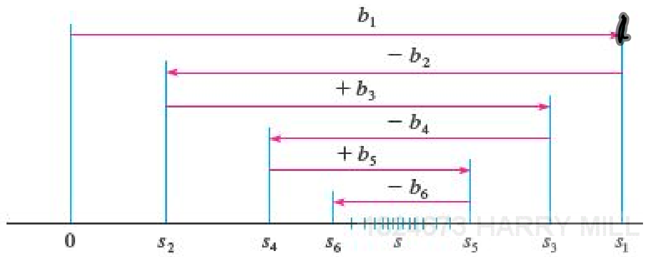


FIGURE 1

We don't always prove theorems, but this is a nice, easy argument that's good to see.

$$s_2 = b_1 - b_2 \geqslant 0 \qquad \text{since } b_2 \leqslant b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geqslant s_2 \qquad \text{since } b_4 \leqslant b_3$$

In general $\quad s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geqslant s_{2n-2} \qquad \text{since } b_{2n} \leqslant b_{2n-1}$

Thus $\quad 0 \leqslant s_2 \leqslant s_4 \leqslant s_6 \leqslant \cdots \leqslant s_{2n} \leqslant \cdots$

But we can also write

By picture, or logic:

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

$$s_{2n} \leqslant b_1$$

Monotonic Sequence Theorem.

Monotone, increasing, bdd above by $b_1 \Rightarrow$ Convergent.

Even-index partial sums converge:

$$\lim_{n \to \infty} s_{2n} = s$$

Odd-index partial sums converge, since

$$\begin{aligned}
\lim_{n \to \infty} s_{2n+1} &= \lim_{n \to \infty} (s_{2n} + b_{2n+1}) \\
&= \lim_{n \to \infty} s_{2n} + \lim_{n \to \infty} b_{2n+1} \\
&= s + 0 \\
&= s
\end{aligned}$$

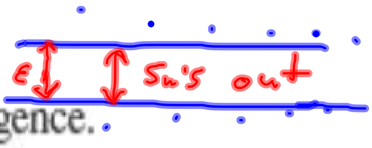
$\to 0$

[see Exercise 80(a) in Section 12.1]

80. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Proof: Ask yo' mama.

Notice that the Theorem does not give us a condition for divergence. So when (i) or (ii) aren't satisfied, we need to look closer. But we can probably do better than the Alternating Series Test, by saying that the series diverges if b_n does NOT approach zero as n approaches infinity (We already have that from our Test for Divergence), if (i) *is* satisfied, because, you know that the b_n's approach a limit by Monotone Sequence (Convergence) Theorem. If they approach a nonzero limit, then the alternation will cause 'em to bounce above and below zero enough to make the SUM bounce around too much.



2–20 Test the series for convergence or divergence.

4. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots$

$$\sum_{k=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^{1/2}} = \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^{1/2}}$$

(i) $\frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \cdots$ so $b_{n+1} \le b_n \ \forall n$.

(ii) $\frac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0$ Converges.

Ramanujan

#12 ought to compare favorably with an alternating harmonic series, but we might have some fun trying to SHOW that the b_n's converge monotonically to zero, although it may take a term or three, before monotone decrease kicks in (a detail you don't want to miss).

$\frac{120}{42}$ $\frac{4200}{840}$ $5! =$

A nice way to get alternating signs is with an $n\pi$ inside a trig function:

$$16.\quad \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$$

$n=1$ $\frac{1}{1} = 1$

$n=2$ 0

$n=3$ $\frac{-1}{3!} = -\frac{1}{6}$

$n=4$ 0

$n=5$ $\frac{1}{120}$

$n=6$ 0

$n=7$ $\frac{-1}{5040}$

$5! = 5\cdot4\cdot3\cdot2\cdot1$

(Our first factorial? $n!$ in the denominator grows pretty fast. How does it compare to exponential growth?)

$$1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040}$$

$$\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}$$

$\boxed{\text{Converges.}}$

$$\frac{e^n}{n!} \xrightarrow{n\to\infty} ?$$

Suppose $\frac{e^n}{n!} > \frac{e^{n+1}}{(n+1)!}$

$$\frac{e^4}{4!} = \frac{e^4}{24}$$

$$\frac{e^5}{5!} = \frac{e^5}{5\cdot24} = \frac{e\cdot e^4}{5\cdot 24} < \frac{e\cdot e^4}{e\cdot24} = \frac{e^4}{24}$$

Suppose $\frac{e^k}{k!} < \frac{e^{k-1}}{(k-1)!}$

Then $\frac{e^{k+1}}{(k+1)!} = \frac{e\cdot e^k}{(k+1)\cdot k!} < ?$

is an idea for showing that $n!$ grows faster than e^n

$e^n < n!$?

$e^{n+1} < (n+1)!$?

ALTERNATING SERIES ESTIMATION THEOREM If $s = \Sigma(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$\text{(i) } 0 \leqslant b_{n+1} \leqslant b_n \quad \text{and} \quad \text{(ii) } \lim_{n\to\infty} b_n = 0$$

then $\displaystyle\sum_{k=n+1}^{\infty}(-1)^{n-1}b_k = |R_n| = |s - s_n| \leqslant b_{n+1}$

has a ceiling of b_{n+1}

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. It follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n| = b_{n+1} \qquad \square$$

21–22 Calculate the first 10 partial sums of the series and graph both the sequence of terms and the sequence of partial sums on the same screen. Estimate the error in using the 10th partial sum to approximate the total sum.

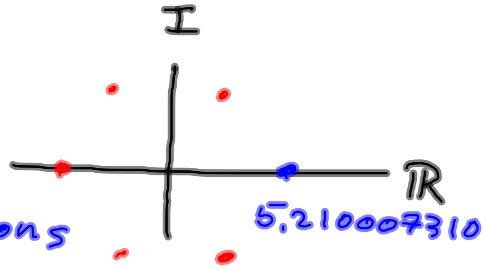
21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}}$ **22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3}}$

I did this a couple ways with technology:
1. With Excel and a pretty efficient method.
2. With Maple and an inefficient method.

I also used an inefficient method with Maple, and spent FAR too much time on it!!!

23–26 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $(|\text{error}| < 0.00005)$



$\sum$ 100000

$solve\left(\frac{1}{x^6} = .00005\right)$

$x^6 = 20{,}000$ has 6 complex solutions

5.210007310, 2.605003655 + 4.511998684 I, −2.605003655 + 4.511998684 I, −5.210007310, −2.605003655 − 4.511998684 I, 2.605003655 − 4.511998684 I

$g(6)$

$\frac{-1}{46656}$

$evalf(g(6))$ $\boxed{-0.00002143347051} = b_{n+1} = b_6$ is less than .00005

$evalf(g(5))$ 0.00006400000000

$g(x) = \frac{-1}{x^6}$

b_k $b_{k+1} \geq |\text{error}|$ So

Error $\leq b_{k+1}$

We found when

b_k = Desired accuracy,

$b_k = g(k)$

Solve $g(k)$ = error desired.

If you get a # between n & $n+1$, choose S_{10},

with error $\leq b_{n+1}$

27–30 Approximate the sum of the series correct to four decimal places.

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$

See 12.5 Worksheet - By hand, you'd churn these out until you found a term that was less than .0001, and take the sum directly before.

32–34 For what values of p is each series convergent?

$\boxed{32.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

34. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

I *think* we're just looking to see what it takes to make the sequence of terms to converge to zero.

35. Show that the series $\Sigma\,(-1)^{n-1}b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?