

How are you guys doing on Chapter 12, so far? 12.1 - 12.3? Is it going fairly quickly? Relative to my schedule, it seems that we're FLYING through the material, but it seems to me in lecture that the pacing is reasonable.

What this tells me is that either I'm reading things wrong, or *we have time to play with some of this stuff, and learn some older stuff BETTER !!!*

§12.1 #39

$$\begin{aligned} a_n &= \left(1 + \frac{r}{n}\right)^n \\ &= \left(1 + \frac{r}{n}\right)^{\frac{n}{2} \cdot 2} \\ &= \left(\left(1 + \frac{r}{n}\right)^{\frac{n}{2}}\right)^2 \\ &\xrightarrow{n \rightarrow \infty} e^2 \end{aligned}$$

$$A(t) = P \left(1 + \frac{r}{m}\right)^{mt} = P \left(1 + \frac{r}{m}\right)^{\frac{m}{r} \cdot r t}$$

$$\begin{aligned} \text{as } m &\rightarrow \infty, \\ A(t) &\rightarrow P e^{rt} \\ &= P \left(1 + \frac{r}{m}\right)^{\frac{m}{r} r t} \\ &\xrightarrow{m \rightarrow \infty} P e^{rt} \end{aligned}$$

$$\lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}} = e$$

See 418, 444

$$\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m = e$$

Period	Balance	$r = \text{rate}$ $m = \text{periods per year}$ $i = \frac{r}{m} = \text{interest rate per period.}$
0	P	
1	$P + P i = P(1+i)$	
2	$P(1+i) + P(1+i)i$ $= P(1+i)[1+i]$ $= P(1+i)^2$	$I = P r t = P r \cdot \frac{1}{m} = P i$ $= \text{Interest per period.}$
3	... $P(1+i)^3$	

$$n \text{ periods: } P(1+i)^n$$

$$\text{Now, with } t = \text{yrs, } n = mt$$

$$\text{This gives } A = P(1+i)^n = P \left(1 + \frac{r}{m}\right)^{mt}$$

$$\text{As } m \rightarrow \text{Big, } A \rightarrow P e^{rt}$$

Compounded Daily:

$$\boxed{P \left(1 + \frac{r}{365}\right)^{365 t}} \approx P e^{rt}$$

Continuous Compounding.

$$\boxed{a_n = n \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} \frac{0}{0}}$$

$\infty \cdot 0$

" = " $\frac{0}{\infty}$ " = " $\frac{0}{0}$

L'Hôpital handles this

$$\stackrel{\text{L'H}}{=} \lim_{n \rightarrow \infty} \frac{-\frac{1}{n^2} \cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

12.4 THE COMPARISON TESTS

THE COMPARISON TEST Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
- (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

So, if it's term-by-term *less* than a convergent series, it converges.
If it's term-by-term *greater* than a *divergent* series, it *diverges*.

Give examples and, per **Note 2**, in the text, a non-example or two, to show the limitations of the (direct) comparison test.

So, $\sum \frac{1}{n^2+1}$ converges because
 $\frac{1}{n^2+1} < \frac{1}{n^2}$ & $\sum \frac{1}{n^2}$ converges.
 & $\sum \frac{1}{n-1}$ DIVERGES $\frac{1}{7} < \frac{1}{6}$
 because $\frac{1}{n-1} > \frac{1}{n}$ & $\sum \frac{1}{n}$ Diverges.

Comparison Test Anxiety:

$\sum \frac{1}{n^2-1}$ No help from comparison test,
 short of gettin' twisted.

$$\frac{1}{n^2-1} < \frac{1}{n^2 - \frac{n^2}{2}} = \frac{1}{\frac{2n^2 - n^2}{2}} = \frac{2}{n^2} \text{ if } n > 2$$

Do not learn
 this trick.

& $\sum \frac{2}{n^2}$ converges.

Much Better to Learn the
 LIMIT COMPARISON TEST!

An Example from the Limit Comparison Test, in the sequel, but a clever student might be able to find a way to apply the Comparison Test, directly.

CourseSmart

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

$$\begin{aligned} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} &> \frac{2n^2}{\sqrt{5 + n^5}} \quad (n \geq 1) > \frac{2n^2}{\sqrt{n^5 + n^5}} = \frac{2n^2}{\sqrt{2n^5}} \\ &= \frac{2n^2}{\sqrt{2} n^{5/2}} = \frac{2}{\sqrt{2} \sqrt{n}} \quad \& \quad \sum \frac{2}{\sqrt{2} \sqrt{n}} \text{ Diverges!} \end{aligned}$$

NOTE 1 Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n , we need verify only that it holds for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

EXAMPLE 2 Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence.

Text uses the term "ultimately." Other terms for this include "eventually" or "for all but finitely many."

The following test will handle the non-examples I provided in the straight-ahead Comparison Test. It'll also clobber anything the Comparison Test will clobber. But the Comparison Test is quicker and cleaner when it *does* work.

THE LIMIT COMPARISON TEST Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Want comparison test, but it's hard to make $\frac{1}{2^n - 1}$ bigger, short of getting twisted.

But $\sum \frac{1}{2^n}$ is nice, it's a geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = a \left(\frac{1}{1-r}\right) = \frac{1}{2} \left(\frac{1}{1-\frac{1}{2}}\right)$$

$$\left. \begin{array}{l} a_n = \frac{1}{2^n - 1} \\ b_n = \frac{1}{2^n} \end{array} \right\} \frac{a_n}{b_n} = \frac{\frac{1}{2^n - 1}}{\frac{1}{2^n}} = \frac{2^n}{2^n - 1}$$

$$= \frac{2^n}{2^n - 1} \xrightarrow{n \rightarrow \infty} 1 = c > 0. \quad \sum \frac{1}{2^n - 1} \text{ converges.}$$

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

Kinda reminds me of

$$\frac{2n^2}{\sqrt{n^5}} = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \quad \sum \frac{2}{n^{1/2}} \text{ diverges!}$$

Expect this will diverge:

$$\frac{a_n}{b_n} = \frac{\frac{2n^2 + 3n}{\sqrt{5 + n^5}}}{\frac{2}{\sqrt{n}}} = \frac{n^2(2 + \frac{3}{n})}{\sqrt{n^5(\frac{5}{n^5} + 1)}} \cdot \frac{\sqrt{n}}{2}$$

$$= \frac{n^2(2 + \frac{3}{n})}{\sqrt{n^5} \sqrt{\frac{5}{n^5} + 1}} \cdot \frac{n^{1/2}}{2} = \frac{\cancel{n^2}(2 + \frac{3}{n})}{2 \cancel{n^2} \sqrt{\frac{5}{n^5} + 1}}$$

$$\xrightarrow{n \rightarrow \infty} \frac{2}{2\sqrt{1}} = 1 \quad \text{Both Diverge!}$$

3-32 Determine whether the series converges or diverges.

5.
$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$$

6.
$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$$

$$12. \sum_{n=0}^{\infty} \frac{1 + \sin n}{10^n}$$

ESTIMATING SUMS

The idea is that you can use the "nicer" series, that you used to compare to your "ugly" series to get a handle on the error in the first n terms of your ugly series.

If you used the comparison test on $s = \sum_{k=1}^{\infty} a_k$,

comparing it to $t = \sum_{k=1}^{\infty} b_k$, which converges, *and has*

all positive terms (or eventually does...), then from the fact that $a_n \leq b_n$, we find that $R_n \leq T_n$, where

$$R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k \quad \text{and} \quad T_n = t - t_n = \sum_{k=n+1}^{\infty} b_k.$$

Since you usually pick a series that's EASIER to compare with, usually the integration estimates we make are EASIER to make for the T_n .

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

$$36. \sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$$

	A	B	C
1	k	a_k	s_k
2	1	0.166666667	0.166666667
3	2	0.074074074	0.240740741
4	3	0.027777778	0.268518519
5	4	0.009876543	0.278395062
6	5	0.003429355	0.281824417
7	6	0.001175779	0.283000196
8	7	0.000400091	0.283400287
9	8	0.000135481	0.283535768
10	9	4.57247E-05	0.283581493
11	10	1.53955E-05	0.283596888
12	11	5.17461E-06	0.283602063
13	12	1.73693E-06	0.2836038
14	13	5.82424E-07	0.283604382
15	14	1.95137E-07	0.283604577
16	15	6.5336E-08	0.283604643
17	16	2.18641E-08	0.283604665
18	17	7.31333E-09	0.283604672
19	18	2.44532E-09	0.283604674
20	19	8.17372E-10	0.283604675
21	20	2.7314E-10	0.283604676
22	21	9.12537E-11	0.283604676
23	22	3.04809E-11	0.283604676
24	23	1.01795E-11	0.283604676
25	24	3.39908E-12	0.283604676
26	25	1.13484E-12	0.283604676
27	26	3.78841E-13	0.283604676
28	27	1.26454E-13	0.283604676
29	28	4.22051E-14	0.283604676
30			

▼ Expression

$\int f dx$ $\int_a^b f dx$ $\sum_{i=k}^n f$

$\prod_{i=k}^n f$ $\frac{d}{dx} f$ $\frac{\partial}{\partial x} f$

$\lim_{x \rightarrow a} f$ a^b $\frac{a}{b}$

a_n a_* \sqrt{a}

$\sqrt[n]{a}$ $a!$ $|a|$

e^a $\ln(a)$ $\log_{10}(a)$

$\log_b(a)$ $\sin(a)$ $\cos(a)$

$\tan(a)$ $\binom{a}{b}$ $f(a)$

$f(a, b)$ $f := a \rightarrow y$

$f := (a, b) \rightarrow z$

$f(x) \Big|_{x=a} \begin{cases} -x & x < a \\ x & x \geq a \end{cases}$

► Units (SI)

► Units (FPS)

▼ Common Symbols

π e i j I ∞

Σ Π \int d \cap \cup

\geq $>$ \nlessgtr \neq \leq $<$

\nlessgtr \nlessgtr α \approx \sim $=$

\neq \equiv \neq \in \notin \subseteq

$$\int_{10}^{\infty} \frac{x}{(x+1) \cdot 3^x} dx$$

`evalf(%)`

$$-\frac{1}{59049} \frac{177147 \operatorname{Ei}(1, 11 \ln(3)) \ln(3) - 1}{\ln(3)}$$

0.00001411402009

$$\int_{10}^{\infty} \frac{1}{3^x} dx$$

`evalf(%)`

$$\frac{1}{59049 \ln(3)}$$

0.00001541498123

This one is more for *my* fun. At worst, it'll be a bonus question on the next test. Speaking of which, how do we want to handle this? We're getting ahead of schedule, here. Might want to stop and do some reviewing before we plunge too deeply into the later material in Chapter 12.

One issue students have with Chapter 12 is that the different tests, especially the Ratio Test -versus- Limit Comparison Test, get confused in their minds... I'd like to try to structure things to avoid that, as best I might.

39. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.