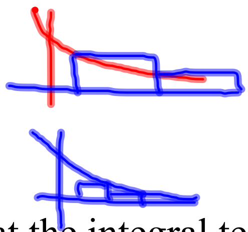
12.3 #s 1–3, 7–12, 15, 20, 30, 32, 35

12.3 THE INTEGRAL TEST AND ESTIMATES OF SUMS

Recall: If f is a positive, decreasing function, then approximating the area under f by Riemann sums is an...



- ...OVERestimate, if you use left endpoints
- ...UNDERestimate if you use right endpoints.

This is a good thing to have in mind as we look at the integral test for convergence and integral techniques for approximating sums that are otherwise difficult to get a handle on. What's *new* is that the roles are reversed, and we're using integrals to estimate infinite sums!!!

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

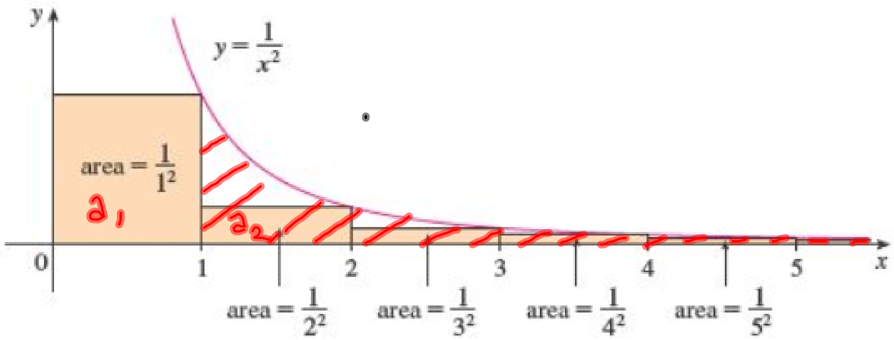


FIGURE 1

$S_n \le 2 \quad \forall n$

$$S=\frac{1}{1^2}+\sum_{n=2}^{\infty}\frac{1}{n^2} < \frac{1}{1^2}+\int_1^{\infty}\frac{1}{x^2}\,dx=2$$

So the sequence of partial sums is bounded above by 2, and since each term is positive, we know that the sequence of partial sums is increasing. By Monotone Convergence Theorem, we know that the series converges.

THE INTEGRAL TEST Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

This could've been shortened-up with a single "if and only if" statement, as the following discussion shows.

$\int_1^{\infty} f$ converges $\Longrightarrow$ $\sum_1^{\infty} a_n$ converges is logically equivalent to

$\sum_1^{\infty} a_n$ diverges $\Longrightarrow$ $\int_1^{\infty} f$ diverges

Nice, but big whup.

Justification for the Integral Test (and insight into later estimates)

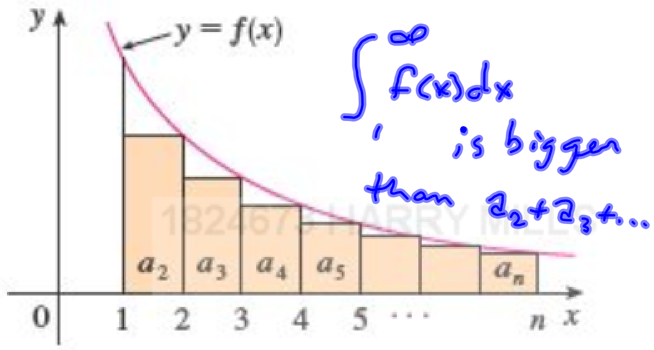


FIGURE 5

The integral is an OVERestimate of the 2-tail = the series without its first term. (just as the right-endpoint Riemann sum would've been an UNDERestimate of the integral in a prequel chapter...)

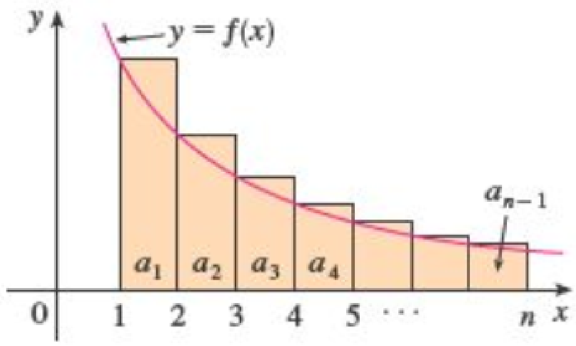


FIGURE 6

The integral is an UNDERestimate of the 1-tail = the entire series. (just as the left-endpoint Riemann sum would've been an OVERestimate of the integral in a prequel chapter...)

Since we only care about whether the "tail" of the series settles down to something finite, these "over- and under-estimates" give us a firm handle on convergence of the series, by what we know about integrals. If the integral converges, so does the the series and if the integral doesn't, neither does the series. Very powerful stuff.

An *immediate* corollary to the Integral Test is the p-test, which flows from what we learned about integrals with indeterminate forms.

$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

$\int_1^\infty \frac{dx}{x^p}$ converges for $p > 1$

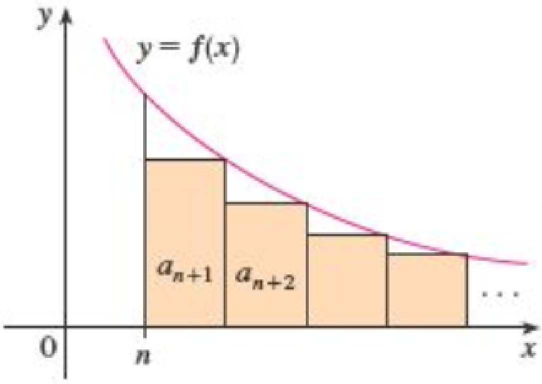
ow diverges.

$\sum \frac{1}{n^{1.2}}$ converges.

ESTIMATING THE SUM OF A SERIES

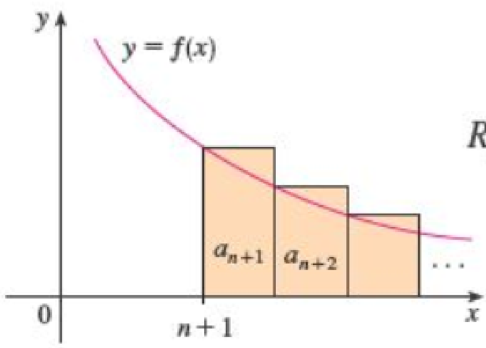
Estimating s by $s_n = a_1 + \cdots + a_n$

Remainder: $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots = (n+1)\text{-tail}$



$$R_n = a_{n+1} + a_{n+2} + \cdots \leqslant \int_n^\infty f(x)\,dx$$

FIGURE 3



$$R_n = a_{n+1} + a_{n+2} + \cdots \geqslant \int_{n+1}^\infty f(x)\,dx$$

FIGURE 4

$\boxed{2}$ **REMAINDER ESTIMATE FOR THE INTEGRAL TEST** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x)\,dx \leqslant R_n \leqslant \int_n^\infty f(x)\,dx$$

See Examples

$\boxed{33.}$ (a) Use the sum of the first 10 terms to estimate the sum of the series $\Sigma_{n=1}^\infty 1/n^2$. How good is this estimate?

(c) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

2 REMAINDER ESTIMATE FOR THE INTEGRAL TEST Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq \boxed{R_n \leq \int_n^{\infty} f(x)\,dx}$$

33. (a) Use the sum of the first 10 terms to estimate the sum of the series $\Sigma_{n=1}^{\infty} 1/n^2$. How good is this estimate? Doggone Good. 5 & 6

(c) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

a) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100}$

$\boxed{s_{10} \approx 1.549767731 \approx s}$

$$R_n \leq \int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_{10}^{t} x^{-2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_{10}^{t}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} + \frac{1}{10}\right] = \boxed{\frac{1}{10} \geq R_n}$$ is not really doggone good.

b) Need $R_n \leq .001$

$$R_n \leq \int_n^{\infty} \frac{1}{x^2}\,dx = \cdots = \frac{1}{n}$$ MAKE $\leq .001 = \frac{1}{1000}$

$\Longrightarrow \boxed{1000 \leq n}$ 1000!?

A BETTER estimate can be obtained by the following observation:

$$s_n + \int_{n+1}^{\infty} \le R_n + s_n \le \int_n^{\infty} + s_n$$

3 $$\boxed{s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx}$$

33. (a) Use the sum of the first 10 terms to estimate the sum of the series $\Sigma_{n=1}^{\infty} 1/n^2$. How good is this estimate?
(b) Improve this estimate using (3) with $n = 10$.
(c) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

#32 Do
R_n both ways

Using this observation, we ought to be able to improve (reduce) our estimate for the number of terms necessary to come within the desired tolerance.

For more on this, see Examples 5 and 6 in text. Example 6 stops when n = 10 turns out to give a good enough estimate. Finding n in general using this tool might be more intractable. We'll see how we do. I'll be flying by the seat of my pants, if n = 10 doesn't work straight outta the gate!

$S_{10} \approx 1.549767731 \approx S$

$$\int_{10}^{\infty} \frac{1}{x^2}dx = \frac{1}{10}$$

$$\int_{11}^{\infty} \frac{1}{x^2}dx = \frac{1}{11}$$

$$S_{10} + \frac{1}{11} \le S \le S_{10} + \frac{1}{10}$$

$$1.549767731 + \frac{1}{11} = 1.640676822 = a$$

$$1.549767731 + \frac{1}{10} = 1.649767731 = b$$

$$\text{Average}: \frac{a+b}{2} = \boxed{1.645222277 \approx S}$$

$$\text{Max ERROR}: \frac{b-a}{2} = .004545454545$$

- How far to get within .001

$$\frac{\frac{1}{11} - \frac{1}{12}}{2} \qquad \frac{\frac{1}{10} - \frac{1}{11}}{2}$$

.003 . . .

$$\frac{\frac{1}{12} - \frac{1}{13}}{2} \quad .0032 \qquad \frac{\frac{1}{13} - \frac{1}{14}}{2} \approx .0027$$

$$\frac{\frac{1}{14} - \frac{1}{15}}{2} \quad \text{Nah} \qquad \frac{1}{15} - \frac{1}{16}$$

$$\frac{\frac{1}{19} - \frac{1}{20}}{2} \approx$$

$\boxed{n = 22}$

$n = 1000$ by just using $R_n \le \int_n^{\infty} f(x)\,dx$