§12.2 Series — Sums of sequences.

$a_1, a_2, \ldots, a_n, \ldots$ is a sequence

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is the corresponding series.

Notation $\sum_{n=1}^{\infty} a_n = \sum a_n$ is OK, as long as context is clear.

Given $S = \sum_{n=1}^{\infty} a_n$,

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is the $n^{\underline{th}}$ partial sum. We say that $\sum_{n=1}^{\infty} a_n$ converges if the sequence of $n^{\underline{th}}$ partial sums $\{S_n\}$ converges, and we write $\sum_{n=1}^{\infty} a_n = L$ if $S_n \xrightarrow{n\to\infty} L$.

Geometric Series

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

$n=1 \quad n=2$

$= \sum_{n=1}^{\infty} ar^{n-1}$ we find the sum.

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$- \quad rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$S_n - rS_n = a - ar^n$

$S_n[1-r] = a - ar^n$

$S_n = \dfrac{a(1-r^n)}{[1-r]}$ is a nice closed-form expression for S_n.

$r^n \to 0$ as $n \to \infty$

Converges if $r \neq 1$

$r \geq 1$ Bad

$r \leq -1$ Bad

$|r| < 1$

§ 12.2 I #s 1, 2, 3, 7–12, 16, 19, 20, 22, 25, 29–31

12.2 II #s 35, 36, 41, 42, 47, 48, 52, 65,

69, 70 → See TB discussion.

$0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1}$

$= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots$

$= 1$

But the geometric series w/ $r = -1$ diverges, so to say it equals zero to begin your discussion is Ludakriss.

Ditch #8

Consider $\{a_n\} = \left\{ \frac{n^2}{11n^2-1} \right\}$

(a) Does $\{a_n\}$ converge? If so, to what?

Yes, to $\boxed{1/11}$ $\frac{1}{11} + \frac{1}{11} + \frac{1}{11} + \frac{1}{11} + \cdots$

(b) Does $\sum_{k=1}^{\infty} a_k$ converge?

No. It constantly grows.

$a_n \not\to 0$ Divergence Test.

Does $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ converge?

$r = -2$, $a_1 = \frac{1}{8}$: $\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)(-2)^{n-1}$

$\boxed{\text{Doesn't}}$

$|r| = 2 \geq 1$

How 'bout $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e \cdot \frac{e^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} e\left(\frac{e}{3}\right)^{n-1}$

Converges to $e\left(\frac{1}{1-\frac{e}{3}}\right) = a\left(\frac{1}{1-r}\right)$

$= e\left(\frac{1}{\frac{3-e}{3}}\right) = \boxed{\frac{3e}{3-e}} = e + \frac{e^2}{3} + \frac{e^3}{9} + \cdots$

$\boxed{\text{Recall:}}$ If $\sum a_n$ & $\sum b_n$ converge to A & B, respectively, then $\sum(a_n+b_n)$ converges to $A+B$, so

$$\sum(a_n+b_n) = \sum a_n + \sum b_n \text{ in this case.}$$

$$0 = 1-1+1-1+1-1+\cdots$$

So, does this apply to $\sum(1-1)$?

No. Neither $\sum 1$ nor $\sum(-1)$ converge.

If $\sum(a_n+b_n)$ converges but $\sum a_n$ diverges, does $\sum b_n$ converge?

No way!

$$\sum \frac{1+2^n}{3^n} = \sum\left(\frac{1}{3^n}+\frac{2^n}{3^n}\right) = \sum\left(\left(\frac{1}{3}\right)^n+\left(\frac{2}{3}\right)^n\right)$$

Does $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converge? Yes $\sum \frac{1}{3}\cdot\left(\frac{1}{3}\right)^{n-1}$

Does $\sum_{n=1}\left(\frac{2}{3}\right)^n$ " ? Yes $\sum_{k=1}^{\infty}\frac{2}{3}\cdot\left(\frac{2}{3}\right)^{n-1}$

So $\sum \frac{1+2^n}{3^n}$ converges!

To $s=$

etc.

$$S_n = \frac{1-r^n}{1-r} = \sum_{k=1}^{n} ar^{k-1}$$

$$S_n = \frac{1-r^{n+1}}{1-r} = \sum_{k=0}^{n} ar^k$$

$a + \cdots\cdots\quad ar^n$

$ar + \cdots\cdots + ar^{n+1}$

$$\sum_{n=1}^{\infty} \ln\left(\frac{3n-1}{5n+2}\right)$$

$$\ln\left(\frac{3n-1}{5n+2}\right) \xrightarrow{n\to\infty} \ln\left(\frac{3}{5}\right) \neq 0$$

Diverges!

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) \qquad \ln\left(1+\frac{1}{n}\right) \xrightarrow{n\to\infty} 0$$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

$$\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = S_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} \left(\ln(k+1) - \ln k\right)$$

No mate

$$= \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \cancel{\ln 4} - \cancel{\ln 3}$$

$$+ \ln 5 - \cancel{\ln 4} + \cdots + \ln(n+1) \cancel{- \ln(n)}$$

No mate.

$$= -\ln 1 + \ln(n+1)$$

$$= \ln(n+1) \xrightarrow{n\to\infty} \infty \quad \text{Diverges!}$$

$\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ is toughie, but partial fractions:

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

$p>1 \Longrightarrow$ convergent

$$\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$$

$A=1,\ B=-1$

$$\sum_{m=2}^{n}\left[\frac{1}{m-1} - \frac{1}{m+1}\right] = \frac{1}{1} - \underline{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \underline{\frac{1}{3}} - \frac{1}{5}$$

$$+ \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{n-1} - \frac{1}{n+1}$$

(S_7 or S_8 might help your intuition.)

$$= \frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \xrightarrow{n \to \infty} \boxed{\frac{3}{2}} = S$$

$+\frac{1}{n-1}$ No $+\frac{1}{n}$ or $+\frac{1}{n+1}$

$$\begin{aligned} .222222\ldots &= x \\ 2.222222\ldots &= 10x \\ \hline 2 &= 9x \\ \frac{2}{9} &= x \end{aligned} \qquad \begin{aligned} .737373\ldots &= x \\ 73.737373\ldots &= 100x \\ \hline 73 &= 99x \\ x &= \frac{73}{99} \end{aligned}$$

$$\frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \cdots = \sum_{n=1}^{\infty} \frac{2}{10}(2/10)^{n-1}$$

$$= \frac{2}{10}\left(\frac{1}{1-\frac{2}{10}}\right) = \frac{2}{10}\,\frac{1}{\frac{10-2}{10}} = \frac{2}{10}\left(\frac{10}{8}\right) =$$

$$2\left(\frac{1}{10}\right) + 2\left(\frac{1}{10}\right)^2 + \cdots +$$