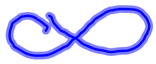
17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.



19. $x = 2\cos\theta, \quad y = \sin 2\theta$

$$\frac{dx}{d\theta} = -2\sin\theta \qquad \therefore \frac{dy}{dx} = \frac{2\cos 2\theta}{-2\sin\theta} = -\frac{\cos 2\theta}{\sin\theta}$$

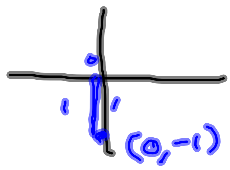
$$\frac{dy}{d\theta} = 2\cos 2\theta$$

Horizontal: $\frac{dy}{dx} = 0 \Longrightarrow \cos(2\theta) = 0$

$2\theta = \cos^{-1}(0)$

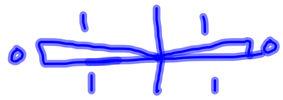
$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

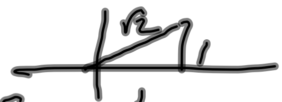


Vertical: $\sin\theta = 0$

$\theta = 0, \pi$



HORIZ: $\theta = \frac{\pi}{4}$: $x = 2\cos\frac{\pi}{4}$



$= 2\cdot\frac{1}{\sqrt{2}} = 2\cdot\frac{\sqrt{2}}{2} = \sqrt{2} = x$

$y = \sin(2\theta) = \sin(2(\frac{\pi}{4})) = \sin(\frac{\pi}{2}) = 1$

$(\sqrt{2}, 1)$ is a pt. where its tangent is horiz.

Etc.

26. Graph the curve $x = \cos t + 2\cos 2t$, $y = \sin t + 2\sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

Might guess $t = \frac{\pi}{2}, \frac{3\pi}{2}$ with Jedi calculator tricks.

Might guess $(x,y) = (-2,0)$ by giving in to the dark side.

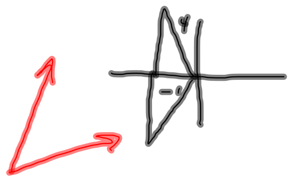
$x = \cos t + 2\cos(2t) = -2$

$\Rightarrow \cos t + 2[2\cos^2 t - 1] = -2$

$4\cos^2 t + \cos t - 2 = -2$

$4\cos^2 + \cos t = 0$

$\cos t\,[4\cos t + 1] = 0$



$t = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ $4\cos t + 1 = 0$

$\cos t = -\frac{1}{4}$

$y = \sin t + 2\sin(2t)$ $\qquad$ $t = \cos^{-1}(-\frac{1}{4})$ is where $x = -2$

$y(\frac{\pi}{2}) = 1 + 0 = 1$

we wanted $y = 0$

$y(\frac{3\pi}{2}) = -1 + 0 = -1$

$y(\cos^{-1}(-\frac{1}{4}))$

$= \sin(\cos^{-1}(-\frac{1}{4})) + 2\sin(2\cos^{-1}(-\frac{1}{4}))$

$= \frac{\sqrt{15}}{4} + 2 \cdot 2\sin\alpha\cos\alpha$

$\alpha = \cos^{-1}(-\frac{1}{4})$

$= \frac{\sqrt{15}}{4} + 4 \cdot \frac{\sqrt{15}}{4} \cdot (-\frac{1}{4}) = \frac{\sqrt{15}}{4} - \frac{\sqrt{15}}{4} = 0$ ✓

Now, plug this MESS into $\frac{dy}{dx}$

$t = \pm \arccos(-\frac{1}{4})$ ✓

$t \in [0, 2\pi]$ – we know The functions are 2π-periodic.

$x = \cos t + 2\cos(2t)$ $\qquad$ $y = \sin t + 2\sin(2t)$

$\frac{dx}{dt} = -\sin t - 4\sin(2t)$

$\frac{dy}{dt} = \cos t + 4\cos(2t)$

$$\frac{dy}{dx} = -\frac{\cos t + 4\cos(2t)}{\sin t + 4\sin(2t)}$$

So $t = \arccos(-\frac{1}{4}) \Rightarrow \cdots \frac{dy}{dx} = -\sqrt{15}$

$t = -\arccos(-\frac{1}{4}) \Rightarrow \cdots \frac{dy}{dx} = \sqrt{15}$

$(x,y) = (-2,0) = (x_1, y_1)$

$y = m(x - x_1) + y_1$

$= \pm\sqrt{15}(x+2) = \pm(\sqrt{15}x + 2\sqrt{15})$

27. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$ in terms of θ. (See Exercise 40 in Section 11.1.)

(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

28. (a) Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ in terms of θ. (Astroids are explored in the Laboratory Project on page 665.)

(b) At what points is the tangent horizontal or vertical?

(c) At what points does the tangent have slope 1 or -1?

29. At what points on the curve $x = 2t^3$, $y = 1 + 4t - t^2$ does the tangent line have slope 1?

30. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

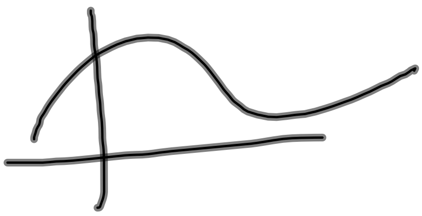
31. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

AREAS

$$\frac{dx}{dt}\,dt = f'(t)\,dt$$

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)\,f'(t)\,dt\right]$$

32. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.



33. Find the area enclosed by the x-axis and the curve $x = 1 + e^t$, $y = t - t^2$.

ARC LENGTH

Recall, from Section 9.1:

3 $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Also, recall, that this model for Arc Length *assumes that the derivative is continuous*.

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 9.1.3 says that if F' is continuous, then

3 $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 2 into Formula 3 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2}$$

Since $dx/dt > 0$, we have

4 $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Even if C can't be expressed in the form $y = F(x)$, Formula 4 is still valid

Basically, it's valid on any stretch where C looks like a function. So you would do a bunch of smaller integrals over those stretches, (provided C doesn't *go vertical for more than an instant at a time*). And you don't have to assume that $dx/dt > 0$, you just formulate L slightly differently on stretches where $dx/dt < 0$...

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

6 THEOREM If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

WARNING: The arc length of the circle described by the parametric equations $x = \cos(2t)$, $y = \sin(2t)$ traverses the circle twice for $t \in [0, 2\pi]$, so our formulation for arc length would give DOUBLE the circumference of the circle!!! ~~So be careful about whether or not $dx/dt > 0$, in particular.~~

So know if your function's periodic & what its period is.

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t - t^2, \quad y = \frac{4}{3}t^{3/2}, \quad 1 \leqslant t \leqslant 2$

69. The **curvature** at a point P of a curve is defined as

$$\kappa = \left| \frac{d\phi}{ds} \right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 14.

(a) For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

(b) By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$, with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$