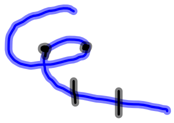
## 11.2 CALCULUS WITH PARAMETRIC CURVES



### TANGENTS

While we can't say necessarily that $y = f(x)$, we *can* find tangents (slopes) of a parametric curve at a given point $(x,y)$ on the curve. The *trick* is to assume that $y = F(x)$, at least in spots, and if it *does*, then we can *do* some things.

$$x = f(t)$$
$$y = g(t)$$

Assume we can eliminate the parameter and get $y = F(x)$.

$\frac{dy}{dx}$ ?

Then : $y = g(t) = F(f(t))$

$$\Rightarrow \frac{dy}{dt} = g'(t) = F'(f(t))f'(t) = \frac{dF}{df}\cdot\frac{df}{dt} = \frac{dF}{dx}\cdot\frac{dx}{dt} = F'(x)f'(t)$$

$\Rightarrow \dfrac{g'(t)}{f'(t)} = F'(x)$ and $F'(x) = \dfrac{dy}{dx}$ gives us the slope of the curve at a point.

This is pretty cool. You just have to compute the derivatives of $y$ and $x$ with respect to $t$ and divide! That's better'n the Quotient Rule!!!

Summary: $x = f(t)$, $y = g(t)$ for some curve $C$.

**1** $$F'(x) = \frac{g'(t)}{f'(t)}$$



**2** $$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0}$$

**1–2** Find $dy/dx$.

**1.** $x = t \sin t, \quad y = t^2 + t$

$$\frac{dx}{dt} = \sin t + t\cos t$$

$$\frac{dy}{dt} = 2t + 1$$

$$\therefore \frac{dy}{dx} = \frac{2t+1}{\sin t + t\cos t}$$

**3–6** Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

**3.** $x = t^4 + 1, \quad y = t^3 + t; \quad t = -1$

$$\frac{dx}{dt} = 4t^3$$

$$\frac{dy}{dt} = 3t^2 + 1$$

$$\therefore \frac{dy}{dx} = \frac{3t^2+1}{4t^3}$$

$$\therefore \left.\frac{dy}{dx}\right|_{t=-1} = \frac{3(-1)^2+1}{4(-1)^3} = \frac{4}{-4} = -1 = m_{tan}$$

Need $(x_1, y_1)$

$x(-1) = (-1)^4 + 1 = 2 = x_1$

$y(-1) = (-1)^3 + (-1) = -2 = y_1$

$(x_1, y_1) = (2, -2)$

$m = -1$

$$y - y_1 = m(x - x_1)$$

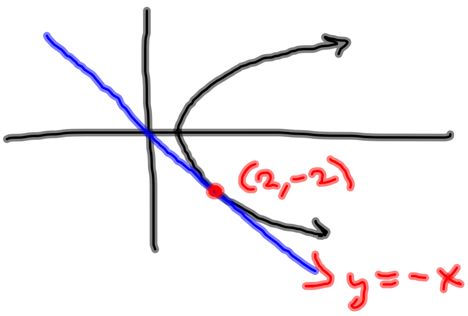
$$y = m(x - x_1) + y_1$$

$$y = -1(x-2) - 2$$

$$y = -x + 2 - 2$$

$$\boxed{y = -x}$$

$x = T$
$y = -T$ } for calculator



**9–10** Find an equation of the tangent(s) to the curve at the given point. Then graph the curve and the tangent(s).

**10.** $x = \cos t + \cos 2t, \quad y = \sin t + \sin 2t; \quad (-1, 1)$

If it crosses itself, there're 2 (or more!) tangent lines to the curve @ that point. That doesn't happen, here.

$$\frac{dx}{dt} = -\sin t - 2\sin(2t) \qquad \therefore \frac{dy}{dx} = \frac{\cos t + 2\cos(2t)}{-\sin t - 2\sin(2t)}$$

$$\frac{dy}{dt} = \cos t + 2\cos(2t)$$

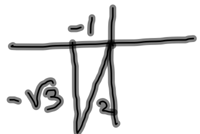
$(x, y) = (-1, 1)$

$x = \cos t + \cos(2t) = -1$

$= \cos t + 2\cos^2 t - 1 = -1$

$\Longrightarrow 2\cos^2 t + \cos t = 0$

$\cos t\,[2\cos t + 1] = 0$



$t = \frac{4\pi}{3}$

$\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

$2\cos t + 1 = 0$

$\cos t = -\frac{1}{2}$

All you need on $[0, 2\pi]$ & $[0, 2\pi]$ is all you need, because $x$ & $y$ are $2\pi$-periodic.

$\pi - \frac{\pi}{3} = \frac{2\pi}{3} = t$

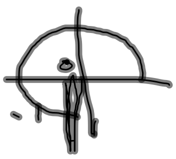
$t = \frac{2\pi}{3}, \frac{4\pi}{3}$

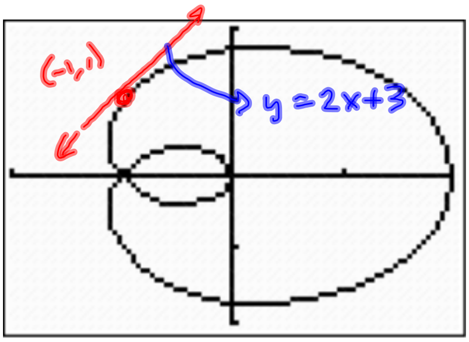
want $y(t) = 1$

$y(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) + \sin(\pi) = 1$

$y(\frac{3\pi}{2}) = \sin(\frac{3\pi}{2}) + \sin(3\pi) = -1$

Nope






$y = m(x - x_1) + y_1$

$= m(x+1) + 1$

$$\boxed{y = 2(x+1) + 1}$$

$y = 2x + 3$

$$\therefore \frac{dy}{dx} = \frac{\cos t + 2\cos(2t)}{-\sin t - 2\sin(2t)} \Longrightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{\cos(\pi/2) + 2\cos(\pi)}{-\sin(\pi/2) - 2\sin(\pi)}$$

$$= \frac{-2}{-1} = 2 = m$$

As we know from Chapter 4, it is also useful to consider $d^2y/dx^2$. This can be found by replacing $y$ by $dy/dx$ in Equation 2:



$$\boxed{2} \qquad \boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Now replace $y$ by $\frac{dy}{dx}$ :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Note that $\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

**11–16** Find $dy/dx$ and $d^2y/dx^2$. For which values of $t$ is the curve concave upward?

**12.** $x = t^3 - 12t, \quad y = t^2 - 1$

$$\frac{dx}{dt} = 3t^2 - 12 \qquad \therefore \frac{dy}{dx} = \frac{2t}{3t^2-12}$$

$$\frac{dy}{dt} = 2t$$

$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2(3t^2-12)-(2t)(6t)}{(3t^2-12)^2}}{3t^2-12}$$

$$= \frac{6t^2-24-12t^2}{(3t^2-12)^3} = \frac{-6t^2-24}{3^3(t^2-4)^3} = \frac{-6(t^2+4)}{3^3(t^2-4)^3}$$

$$= \frac{-2(t^2+4)}{9(t-2)^3(t+2)^3}$$ want $> 0$

Sign pattern for $\frac{d^2y}{dx^2}$

| | $-$ | | $+$ | | $-$ |
|---|---|---|---|---|---|
| | No | $-2$ | Yes | $2$ | No |

$\boxed{\text{It's concave up } \forall\, x \in (-2,2)}$

§11.2 I #s 1, 2, 4, 5, 9, 11, 18, 25, 29