§8.3 #s 4, 8, 16, 17, 21, 30, 34, 35, 42

8.3 TRIGONOMETRIC SUBSTITUTION

#17: $7 = (\sqrt{7})^2$

Recall: Substitution Rule

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Today, we do this:

$$\int f(u)\, du = \int f(g(x))\, g'(x)\, dx$$

$$\boxed{\sin^{-1} x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\, dx = \int_0^1 \frac{du}{1+u^2}$$

$u = \sin x \longrightarrow du = \cos x\, dx$

$\sin(0) = 0$

$\sin\left(\frac{\pi}{2}\right) = 1$

Today: Going from $\int_{0=u}^{1=u} \frac{du}{1+u^2}$ to $\int_{0}^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\, dx$

Let $u = \sin x$

$du = \cos x\, dx$

$u = 0 = \sin x$

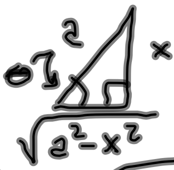
$0 = \sin^{-1}(0) = x$

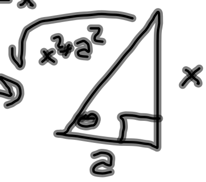
$u = 1 \Longrightarrow$

$x = \sin^{-1}(1) = \frac{\pi}{2}$

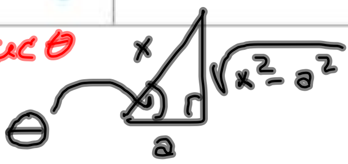
$\int_{\sin^{-1}(0)}^{\sin^{-1}(1)} \frac{\cos x\, dx}{1+\sin^2 x}$ is how we'd go the other way.

Expression	Substitution	Identity
$\boxed{\sqrt{a^2 - x^2}}$	$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad \frac{x}{a} = \sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\quad \frac{x}{a} = \tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$





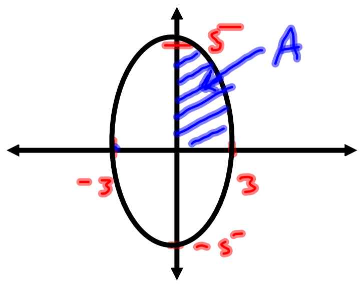
$\frac{x}{a} = \sec\theta$



$\boxed{E}$ Area of the ellipse

Area = 4A

$$\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$$



Solve for y!

$$5^2x^2 + 3^2y^2 = 5^2\cdot 3^2$$

$$3^2y^2 = 5^2\cdot 3^2 - 5^2x^2$$

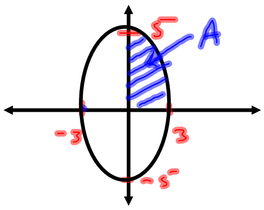
$$\sqrt{a^2 - x^2}$$

$$y^2 = \frac{5^2\cdot 3^2}{3^2} - \frac{5^2}{3^2}x^2$$

$$y = \pm\sqrt{5^2 - \frac{5^2}{3^2}x^2} = \pm\sqrt{\frac{3^2\cdot 5^2}{3^2} - \frac{5^2}{3^2}x^2}$$

$$= \pm\sqrt{\frac{5^2}{3^2}\left(3^2 - x^2\right)} = \pm\frac{5}{3}\sqrt{3^2 - x^2}$$

Top piece: $\frac{5}{3}\sqrt{3^2 - x^2} = y$



$\frac{5}{3}\sqrt{3^2-x^2} = y$

$$\text{Area} = 4\int_0^3 \frac{5}{3}\sqrt{3^2-x^2}\,dx$$

$$= \frac{(4)(5)}{3}\int_0^3 \sqrt{3^2-x^2}\,dx$$

Let $x = 3\sin\theta$
$dx = 3\cos\theta\,d\theta$
$x=0 \Rightarrow 3\sin\theta = 0$
$\sin\theta = 0$
$\theta = 0$
$x = \Rightarrow 3\sin\theta = 3$
$\Rightarrow \sin\theta = 1$
$\Rightarrow \theta = \frac{\pi}{2}$

This gives: $\frac{(4)(5)}{3}\int_0^{\frac{\pi}{2}}\left(\sqrt{3^2-(3\sin\theta)^2}\right)(3\cos\theta\,d\theta)$

Scratch: $3^2-(3\sin\theta)^2 = 3^2 - 3^2\sin^2\theta = 3^2(1-\sin^2\theta)$
$= 3^2\cos^2\theta$

$$= \frac{(4)(5)}{\cancel{3}}\int_0^{\frac{\pi}{2}} \sqrt{3^2\cos^2\theta}\cdot\cancel{3}\cos\theta\,d\theta$$

NOTE: $|\cos\theta| = \cos\theta$
$\forall\, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= (4)(5)\int_0^{\frac{\pi}{2}} 3\cos\theta\,\cos\theta\,d\theta$$

$$= (4)(5)(3)\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= (4)(5)(3)\int_0^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2\theta]\,d\theta$$

$$= \frac{(4)(5)(3)}{2}\left[\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\,2d\theta\right]$$

$$= (2)(5)(3)\left[\frac{\pi}{2} - 0 + \frac{1}{2}\left[\sin(2\theta)\right]_0^{\frac{\pi}{2}}\right]$$

$$= (\cancel{2})(5)(3)\left[\frac{\pi}{\cancel{2}} + \frac{1}{2}[0-0]\right]$$

$$= (5)(3)\pi$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Area $= ab\pi$

$$= \frac{(4)(5)(3)}{2}\left[\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, 2\,d\theta\right]$$

$$= (2)(5)(3)\left[\theta\Big]_0^{\frac{\pi}{2}} + \underline{\frac{1}{2}\sin(2\theta)}\right]_0^{\frac{\pi}{2}}$$

want the expression in x.

$$= (2)(5)(3)\left\{ \sin^{-1}\frac{x}{3}\right]_0^3 + \frac{1}{2}$$

$x = 3\sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$x = 3\sin\theta,$

so $\sin(2\theta) = ?$

Hold this thought!

24. $\displaystyle\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{dt}{\sqrt{(t-3)^2+2^2}} = \int \frac{du}{\sqrt{u^2+2^2}}$

$t^2-6t+13$

$= t^2-6t+3^2-9+13$

$= (t-3)^2+4$

$= (t-3)^2+2^2$

Let $u = t-3$
$du = dt$

$\sqrt{x^2} = |x|$

$u = 2\tan\theta$
$du = 2\sec^2\theta\, d\theta$

gives $\displaystyle\int \frac{2\sec^2\theta\, d\theta}{\sqrt{2^2\sec^2\theta}}$

$u^2+2^2 = 2^2\tan^2\theta + 2^2$
$= 2^2(\tan^2\theta+1)$

$\displaystyle= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta}$

$\displaystyle= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

Pf: $\displaystyle\int \sec\theta\left(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\right)d\theta$, etc.

$u = 2\tan\theta \longrightarrow \frac{u}{2} = \tan\theta$

$\displaystyle\ln\left|\frac{\sqrt{u^2+2^2}}{2} + \frac{u}{2}\right| + C_1$

$\displaystyle= \ln\left|\frac{\sqrt{(t-3)^2+2^2}}{2} + \frac{t-3}{2}\right| + C_1$

$= \ln\left|\sqrt{t^2-6x+13} + t-3\right| + C$, where $C = C_1 + \ln\frac{1}{2}$

$\ln\left(\frac{1}{2}(A+B)\right) = \ln\frac{1}{2} + \ln(A+B)$

just a constant

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.