7.8 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

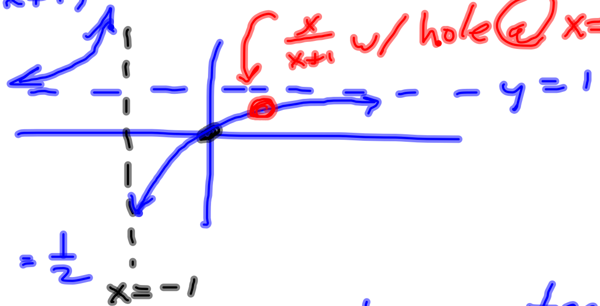
$f(x) \to 0$ and $g(x) \to 0$ as $x \to a$,

indeterminate form of type $\frac{0}{0}$.

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{1}{1+1} = \frac{1}{2}$$

$\frac{0}{0}$

$\frac{x}{x+1}$ w/ hole @ $x=1$



$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1}{2}$$

$= \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{x^2}{2x^2}$, 'cuz -1 & $+1$ are teeny.

1 $\lim_{x \to 1} \frac{\ln x}{x - 1} = \frac{0}{0}$

2 $\lim_{x \to \infty} \frac{\ln x}{x - 1} = \frac{\infty}{\infty}$

$\frac{\infty}{\infty}$

Other indeterminate forms:

$0 \cdot \infty$ *

$\infty - \infty$ *

0^0 *

∞^0 *

1^∞ *

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

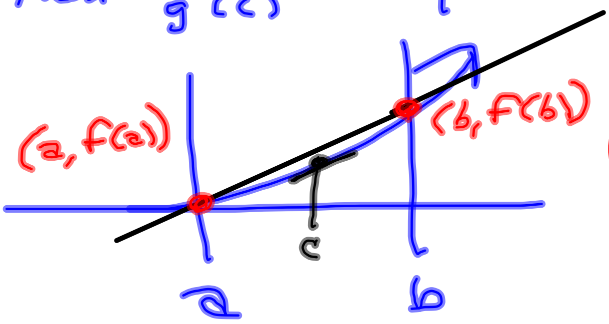
Figure 1 is an attempt to help your intuition, somewhat. It kind of makes sense that if numerator and denominator both approach zero, the actual limit of the quotient probably is controlled by how *fast* each is approaching zero. In other words, their slopes at that limiting value can easily be imagined to have some sort of effect on the limit of the quotient.

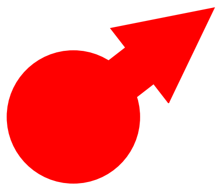
3 **CAUCHY'S MEAN VALUE THEOREM** Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

Generalized MVT

NOTE: If $g(x) = x$, then $\dfrac{f'(c)}{g'(c)} = \dfrac{f'(c)}{1} = f'(c) = \dfrac{f(b) - f(a)}{b - a}$





James & Joel did this for §7.6 #21

Claim: $\frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

Proof: $y = \csc^{-1}x \Longrightarrow \csc y = x$ (*)

Differentiating w.r.t. x: $-\csc y \cot y \frac{dy}{dx} = 1$

$$\Longrightarrow \frac{dy}{dx} = -\frac{1}{\csc y \cot y}$$

$$= -\frac{1}{x\cot y}, \text{ by } (*)$$

Now, we use pythagorean identity to express $\cot y$ in terms of x:

$\csc^2 y = \cot^2 y + 1 \Longrightarrow \cot^2 y = \csc^2 y - 1$

$$\Longrightarrow \cot y = \pm\sqrt{\csc^2 y - 1}.$$

Now, looking at set of y-values possible, we see that they come from the 1st & 3rd quadrants, by the original $y = \csc^{-1}x$, so $\cot y \geq 0$, hence $\cot y = +\sqrt{\csc^2 y - 1}$

Finally, $\csc^2 y = x^2$, since $\csc y = x$,

and so $\frac{d}{dx}\left[\csc^{-1}x\right] = -\frac{1}{x\sqrt{x^2-1}}$ ■