7.7 HYPERBOLIC FUNCTIONS

7.7 #s 1 – 6, 8, 10, 18, **19***, 20, 24a, 26, 28c, 34, 40, 46, **52** (Class project - show all steps, everyone get involved.)
#s 1 – 6 (and 'most all the assignment), think in terms of $\dfrac{e^x - e^x}{2}$

Also remember that an often-practical way to evaluate an inverse hyperbolic/trigonometric function is to solve a hyperbolic/trigonometric equation, and check your domains (off a cheatsheet). We will collaborate on a cheatsheet for the class to use.
#s 11, 12 are not assigned, but look like good cheatsheet material.
#19 is a nice bonus problem. Answer's in the back. Looks like it oughta have a nice, clean induction proof.

DEFINITION OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$f(-x) = -f(x)$

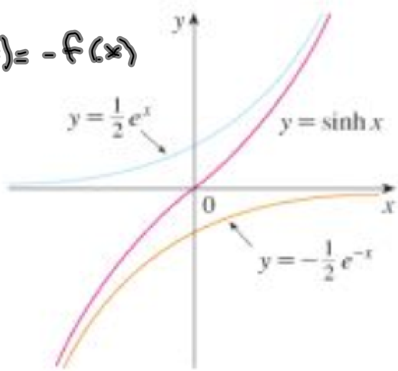


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$f(-x) = f(x)$

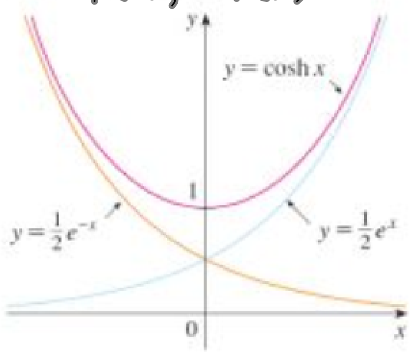
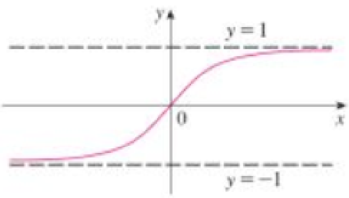


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$



$$-1 < \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x < 1$$

$$|\coth x| > 1$$

FIGURE 3
$y = \tanh x$

HYPERBOLIC IDENTITIES

$\sinh(-x) = -\sinh x$ $\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta + 2\pi) = \sin\theta$ $\cos(\theta + 2\pi) = \cos\theta$

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

1 DERIVATIVES OF HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

2

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

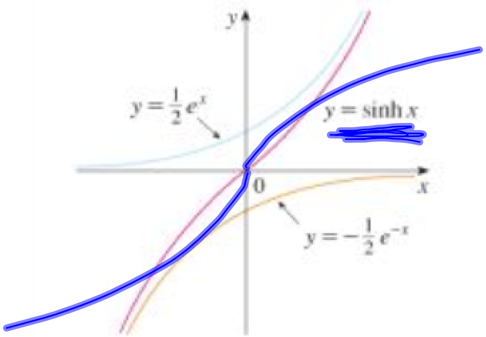


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$\sinh^{-1}(1)$
$\sinh x = 1$

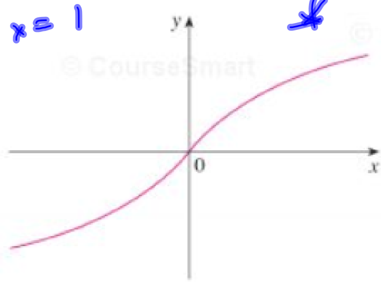


FIGURE 8 $y = \sinh^{-1}x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

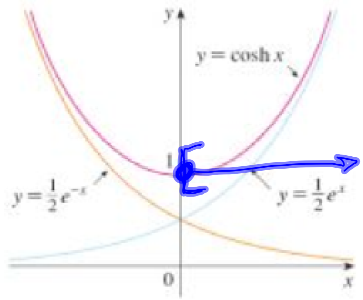


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

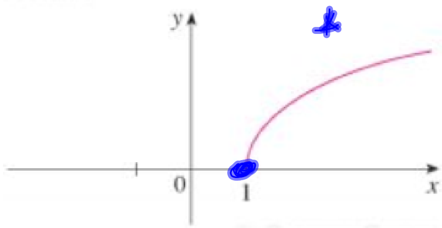


FIGURE 9 $y = \cosh^{-1}x$
domain $= [1, \infty)$ range $= [0, \infty)$

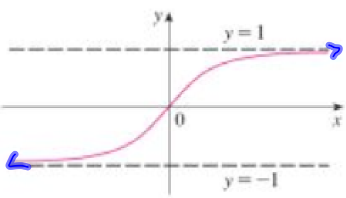


FIGURE 3
$y = \tanh x$

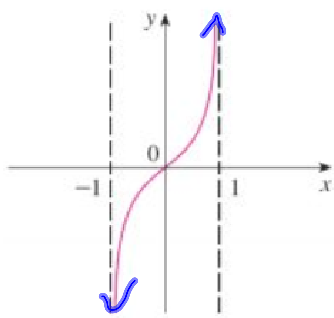


FIGURE 10 $y = \tanh^{-1}x$
domain $= (-1, 1)$ range $= \mathbb{R}$

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3	$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$	$x \in \mathbb{R}$
4	$\cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \geqslant 1$
5	$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$-1 < x < 1$

The proof of the first of these identities is given in Example 3. This is worth going over, with a slightly different way of getting started.

$\boxed{\text{Pf of 3}}$

$y = \sinh^{-1} x$

$x = \sinh y$ solve for y

$\sinh y = x$

$\frac{e^y - e^{-y}}{2} = x$

$e^y - e^{-y} = 2x$

$e^y - 2x - e^{-y} = 0$

$(e^y)^2 - 2xe^y - 1 = 0$

$u^2 - 2xu - 1 = 0$

$u = e^y$

$a = 1, b = -2x$

$c = -1$

$b^2 - 4ac =$

$(-2x)^2 - 4(1)(-1)$

$= 4x^2 + 4$

$e^y = u = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{2x \pm \sqrt{4x^2+4}}{2}$

$0 < e^y = e^y = e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$

$\Rightarrow e^y = x + \sqrt{x^2+1}$

$y = \ln\left(x + \sqrt{x^2+1}\right) = \sinh^{-1}x$

6 DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\boxed{\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}}$$

See Example 4. It's the sort of argument one might be expected to make for the derivative of the inverse hyperbolic cosine.

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

$y = \sin^{-1}x$

$\sin y = x$

differentiate w.r.t. x

To remember this or to help keep it straight, look at the definitions of $\sinh x$ and $\cosh x$ and notice that $-1 < \tanh x < 1$, whereas $|\coth x| > 1$.

Something you won't see in the textbook:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

To read more on this, Google "exponential form of sine" and follow the *Wolfram* link.

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

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$P(n)$: $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$ $\forall n \in \mathbb{N}$

Let $S' = \{n \in \mathbb{N} \mid$ the statement $P(n)$ holds$\}$

Prove that $1 \in S'$

Then show that if $k \in S'$, then $k+1 \in S'$, also.

$\therefore S' = \mathbb{N}$ & $P(n)$ holds $\forall n \in \mathbb{N}$.

Example

$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$: $P(n)$

Let $S' = \{n \in \mathbb{N} \mid P(n)$ holds$\}$

Clearly $1 \in S'$, since $1 = \frac{1(1+1)}{2} = 1$

Suppose $k \in S'$, $k \geq 1$.

Then $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$

Consider $\underbrace{1 + 2 + 3 + \cdots + k}$ $+ k+1$ WANT $\frac{(k+1)(k+2)}{2}$

$$= \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$\frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

$\Longrightarrow k+1 \in S'$

$\therefore S' = \mathbb{N}$, by PMI

$(\cosh x + \sinh x)^1 = \cosh x + \sinh x$ ✓

$(\cosh x + \sinh x)^2 = \cosh^2 x + 2\cosh x \sinh x + \sinh^2 x$

$$\left(\frac{e^x+e^{-x}}{2}\right)^2 + 2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right) + \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$\frac{e^{2x}+2+e^{-2x}}{4} + 2\left(\frac{e^{2x}-e^{-2x}}{4}\right) + \frac{e^{2x}-2+e^{-2x}}{4}$$

$$\frac{e^{2x}+e^{-2x}+2e^{2x}-2e^{-2x}+e^{2x}+e^{-2x}}{4}$$

$$\frac{4e^{2x}}{4} = \boxed{\frac{2e^{2x}}{2}} = \frac{e^{2x}+e^{2x}}{2}$$

$$= \frac{e^{2x}+e^{-2x}}{2} + \frac{e^{2x}-e^{-2x}}{2}$$

$$= \cosh(2x) + \sinh(2x)$$