

§ 12.4 #35

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

$\frac{1}{n} > 0 \quad \forall n \in \mathbb{N}$
 But $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Note

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

Then

$$R_{10} \leq$$

$$\sum_{n=11}^{\infty} \frac{1}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+11}} = \sum_{n=0}^{\infty} \frac{1}{2^{11}} \cdot \frac{1}{2^n}$$

$$= \frac{1}{2^{11}} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2^{11}} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{11}} \cdot \frac{1}{\frac{1}{2}}$$

$$= \frac{1}{2^{11}} \cdot \frac{2}{1} = \frac{1}{2^{10}}$$

$$R_{10} \leq T_{10} =$$

$$S = S_n + R_n = \sum_{k=1}^n a_k + \sum_{k=n+1}^{\infty} a_k$$

$$= S_{10} + R_{10}$$

$$= \sum_{k=1}^{10} a_k + \sum_{k=11}^{\infty} a_k$$

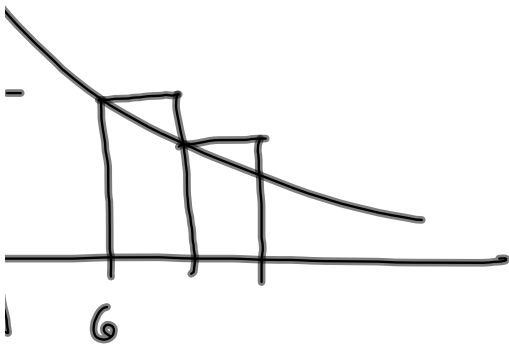
a_n decreasing $a_n = f(n)$

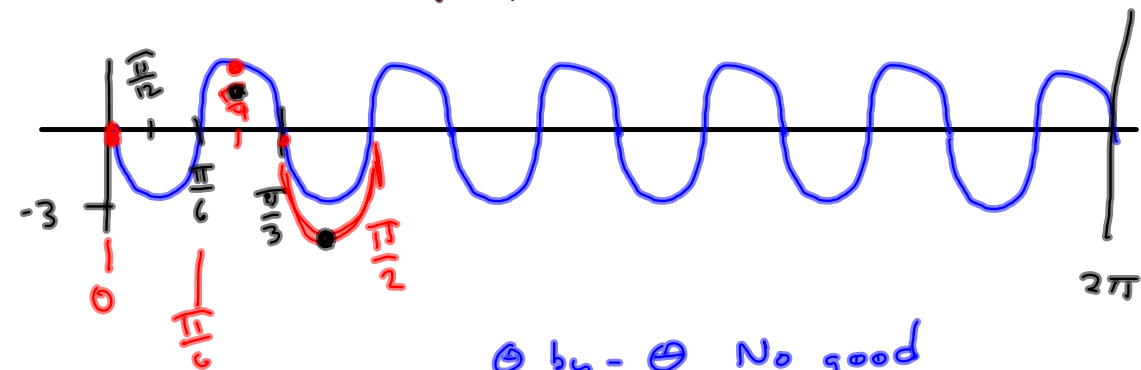
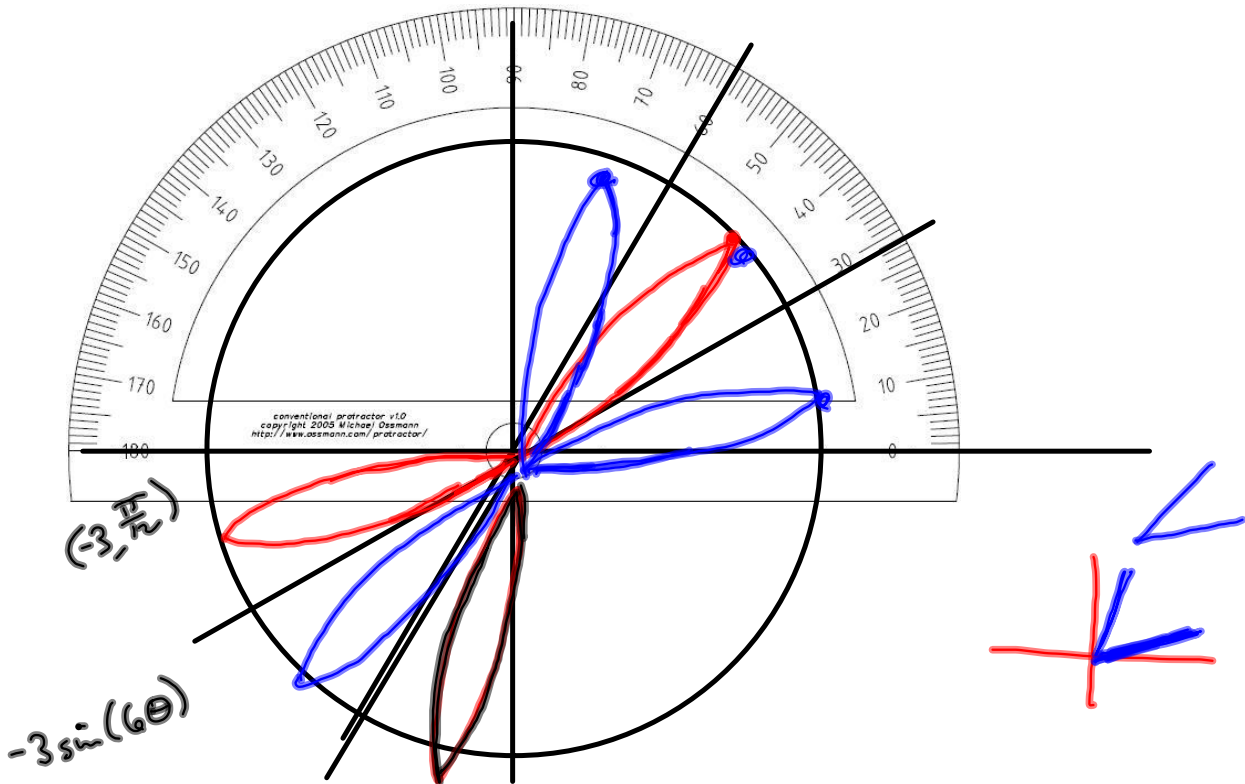
$$S = S_n + R_n$$

$$= S_{10} + R_{11}$$

$$S_{10} + \int_{11}^{\infty} f(x) dx \leq S_{10} + R_{10} \leq S_{10} + \int_{10}^{\infty} f(x) dx$$

Conservative estimate of the error





$r = -3 \sin(6\theta)$

θ by $-\theta$ No good
 r by $-r$ "
 θ by $\theta + \pi$: Polar symmetry
 $\sin(6(\theta + \pi))$
 $= \sin(6\theta + 6\pi)$
 $= \sin(6\theta) \cos(6\pi) + \cos(6\theta) \sin(6\pi)$
 $= \sin(6\theta)$
 $= \sin(6(\theta + \pi))$

θ by $\pi - \theta$: $\sin(6(\pi - \theta)) = \sin(6\pi - 6\theta)$
 $= \sin(6\pi) \cos(-6\theta) + \cos(6\pi) (\sin(-6\theta))$
 $= \sin(-6\theta)$ No help.

Evaluate $\int \arctan(x) dx$ with power series.

Find its radius of convergence & its interval of convergence.

Integrals have a smoothing effect.

$f(x) = \arctan(x)$

$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

What's its radius of convergence?

Need $x^2 < 1$
 $|x| < 1$ means $|x| < 1$
 $-1 < x < 1$ $x < 1$ and $x > -1$

Try $x=1$: $\sum_{n=0}^{\infty} (-1)^n \cdot 1^{2n}$
 $x=-1$: same deal

$x = \pm 1$ Bad.
 I.C. = $(-1, 1)$

$f(x) = \arctan(x) = \int \frac{dx}{x^2+1}$

$|x^3| < 1$
 $|x|^3 < 1$
 $|x| < 1$
 $= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$

$R = 1$

$\left| \frac{x^{(n+1)+1}}{2(n+1)+1} \cdot \frac{2n+1}{x^{2n+1}} \right| = \left| \frac{x^2 \cdot (2n+1)}{2n+3} \right| \xrightarrow{n \rightarrow \infty} |x^2| < 1$

At $x=1$: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1}$
 $x=-1$: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{-1}{2n+1}$
 Both converge, by alternating series test.
 $\frac{x^2}{9} < 1$
 $\frac{x}{3}$

So $R = 1$
 I.C. = $[-1, 1]$

$\int \arctan(x) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)}$
 converges even more strongly than $f(x)$

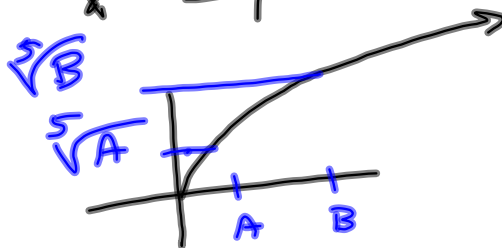
$|x| = 1$ means

$$x = +1 \text{ or } x = -1$$

Does NOT mean that

$|x| < 1$ implies

$$x < \pm 1$$



$A < B$ means

$$\sqrt[A]{A} < \sqrt[B]{B}$$

Pretty neat stuff. Just keep in mind, we've been *assuming* that f actually *has* a power series representation.

As our first requirement for the existence of such a representation, for this method to work, **clearly f must have derivatives of all orders.** (necessary, but not sufficient).

T_n is the n^{th} degree Taylor Polynomial. You'd like to think that as n grows large, that T_n gives a decent approximation for f . Follow the link to TEC, below for an illustration:

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i$$

In general, this representation "works" if T_n converges to f . So to see if such a thing is happening, we basically have to show that the remainder, R_n converges to zero in the following:

$$f(x) = \lim_{n \rightarrow \infty} T_n(x)$$

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

8 THEOREM If $f(x) = T_n(x) + R_n(x)$, where T_n is the n th-degree Taylor polynomial of f at a and

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

$$\sum_{k=0}^n c_k x^k + \sum_{k=n+1}^{\infty} c_k x^k$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

$$T_n + R_n$$

Taylor's :

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

A tool for showing that R_n converges to zero (or for deciding how large n must be in order for T_n to be sufficiently close to f ...

9 TAYLOR'S INEQUALITY If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d$$

How many terms does it take to get $\sin(x)$ accurate to within ± 0.00001

Alternate: want 5th decimal place accurate.

On $[-\pi, \pi]$

Notice that $|\sin(x)| \leq 1$
 $|\cos(x)| \leq 1$

$$\sin(x) = f(x)$$

That's your M

$$\cos(x) = f'(x)$$

Need $\frac{1}{(n+1)!} |2\pi|^{n+1} < 0.00005$

$$\vdots \quad \boxed{a=0}$$

$$d = 2\pi$$

$$\begin{aligned} n=0: & \frac{1}{1} \cdot 2\pi \\ n=1: & \frac{1}{2} (2\pi)^2 \\ n=2: & \frac{1}{3!} (2\pi)^3 \\ n=3: & \frac{1}{4!} (2\pi)^4 \\ n=4: & \frac{1}{5!} (2\pi)^5 \\ n=5: & \frac{1}{6!} (2\pi)^6 \\ n=6: & \frac{1}{7!} (2\pi)^7 \end{aligned}$$

Ouch! Somewhere between 20 & 25 terms?

That's why, in practice, you'd pick a convenient (maybe) value of a to the inputs you'd be using

If you knew all inputs were between $-\frac{\pi}{4}$ & $\frac{\pi}{4}$ This becomes $\frac{1}{(n+1)!} \left(\frac{\pi}{2}\right)^{n+1}$

$$\left| \frac{n+1}{6^{n-1}} x^{4n+1} \right| = \frac{n+1}{6^{n-1}} |x|^{4n+1}$$

if $n > \frac{1}{6}$

If $n = \frac{1}{6}$ bleaching

If $n < \frac{1}{6}$

$$= \frac{|n+1| |x|^{4n+1}}{1-6^n}$$

$$|xy| = |x| |y|$$

$$|6^{n-1}| = -(6^{n-1})$$

$$\text{if } 6^{n-1} < 0$$

So finding bounds on derivatives appears to be on the table.

A useful fact that we no longer have to justify every time we use it is the following:

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$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

The Binomial Coefficient, also known as combinations of n things taken k at a time, is generalized quite a bit in this section, for those of you who've already been exposed to combinations, permutations, and the Binomial Theorem in previous courses.

This is discussed in Example 8.

[17] THE BINOMIAL SERIES If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

What happens when k is a positive integer?

How would you BUILD the power series expansion for this?

An example that bootstraps off the binomial expansion, rather than just using the basic algorithm is given by:

EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2} \left(1-\frac{x}{4}\right)^{-1/2}$$

Now just apply the binomial expansion to the above. I never would've thought of this as a student.

A table of the standard Maclaurin series. Probably good material for a cheat sheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad R = \infty$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \quad R = 1$$

Applications of this theory include:

- Approximating a function by using its Taylor polynomial (Early calculators used Taylor polynomials to do 'most all their trig. calculations). ✓
- Approximating a definite integral by integrating the first handful of terms of the Taylor series corresponding to its integrand. ✓
- Evaluating Limits by expressing all or part of the expression in terms of a series.

$f(x) = \int_0^x \frac{1}{1+t^5} dt$ can be thought of as approximating a function. We plugged in $x=.2$ in the 11/18/11 notes for 12.9.

We also built a power series for $\cos(x)$, centered at $a=0$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Using Taylor's Series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \underbrace{f(a) + f'(a)(x-a)} + \frac{f''(a)}{2} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$