§ 12.4 #35

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

$\frac{1}{n} > 0 \quad \forall n \in \mathbb{N}$

But $\lim_{n\to\infty} \frac{1}{n} \geq 0$

Note

$$\frac{1}{2^n+1} < \frac{1}{2^n} =$$

Then $$R_{10} \leq \sum_{n=11}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+11}} = \sum_{n=0}^{\infty} \frac{1}{2^{11}} \cdot \frac{1}{2^n}$$

$$= \frac{1}{2^{11}} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2^{11}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{11}} \cdot \frac{1}{\frac{3}{2}}$$

$$= \frac{1}{2^{11}} \cdot \frac{2}{3} = \frac{1}{2^{10} \cdot 3}$$

$$R_{10} \leq T_{10} =$$

$$S = S_n + R_n = \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty}$$

$$= S_{10} + R_{10}$$

$$= \sum_{k=1}^{10} a_k + \sum_{k=11}^{\infty} a_k$$

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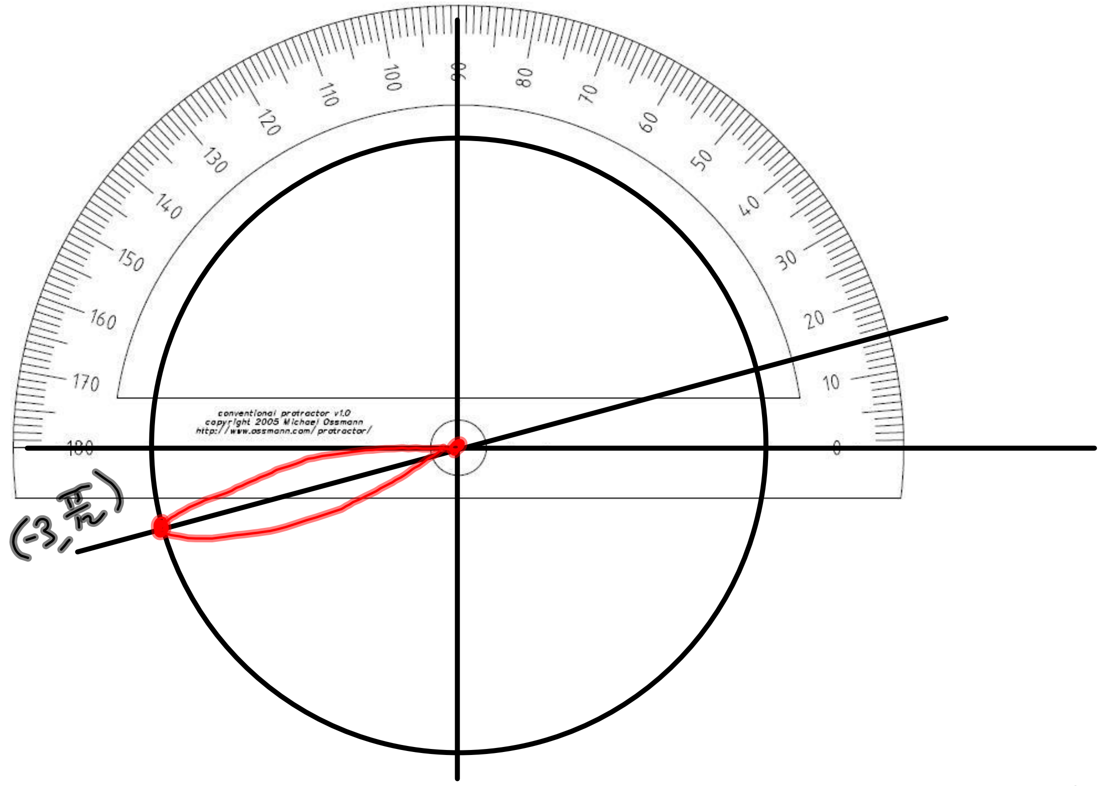
a_n decreasing $\quad a_n = f(n)$

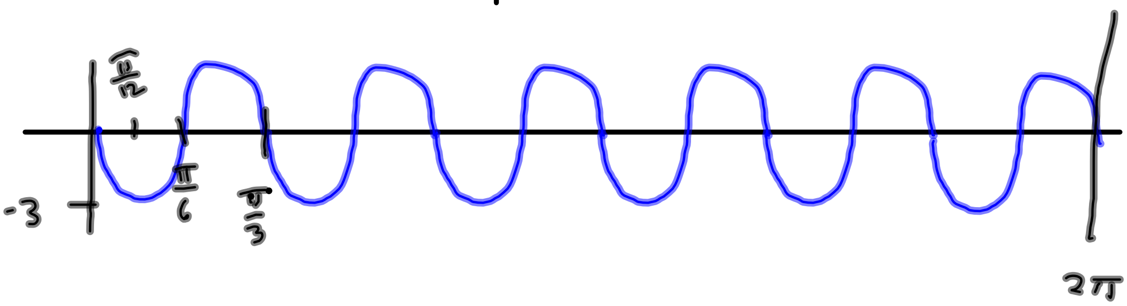
$$S = S_n + R_n$$

$$= S_{10} + R_{11}$$

$$S_{10} + \int_{11}^{\infty} f(x)\,dx \leq S_{10} + R_{10} \leq S_{10} + \int_{10}^{\infty} f(x)\,dx$$

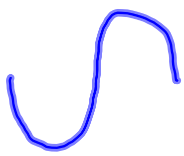
Conservative estimate of the error





$r = -3\sin(6\theta)$ $T = \frac{2\pi}{6} = \frac{\pi}{3}$

$r = 3\cos(6\theta)$ graded



Pretty neat stuff. Just keep in mind, we've been *assuming* that f actually *has* a power series representation.

As our first requirement for the existence of such a representation, for this method to work, **clearly f must have derivatives of all orders**. (necessary, but not sufficient).

T_n is the n^{th} degree Taylor Polynomial. You'd like to think that as n grows large, that T_n gives a decent approximation for f. Follow the link to TEC, below for an illustration:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

In general, this representation "works" if T_n converges to f. So to see if such a thing is happening, we basically have to show that the remainder, R_n converges to zero in the following:

$$f(x) = \lim_{n\to\infty} T_n(x)$$

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

8 THEOREM If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

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A tool for showing that R_n converges to zero (or for deciding how large n must be in order for T_n to be sufficiently close to f...

9 TAYLOR'S INEQUALITY If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leq d$$

So finding bounds on derivatives appears to be on the table.

A useful fact that we no longer have to justify every time we use it is the following:

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$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

The Binomial Coefficient, also known as combinations of n things taken k at a time, is generalized quite a bit in this section, for those of you who've already been exposed to combinations, permutations, and the Binomial Theorem in previous courses.

This is discussed in Example 8.

17 THE BINOMIAL SERIES If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

What happens when k is a positive integer?

How would you BUILD the power series expansion for this?

An example that bootstraps off the binomial expansion, rather than just using the basic algorithm is given by:

V EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4 - x}}$ and its radius of convergence.

SOLUTION We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4 - x}} = \frac{1}{\sqrt{4\left(1 - \dfrac{x}{4}\right)}} = \frac{1}{2\sqrt{1 - \dfrac{x}{4}}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2}$$

Now just apply the binomial expansion to the above. I never would've thought of this as a student.

A table of the standard Maclaurin series. Probably good material for a cheat sheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Applications of this theory include:

• Approximating a function by using its Taylor polynomial (Early calculators used Taylor polynomials to do 'most all their trig. calculations).
• Approximating a definite integral by integrating the first handful of terms of the Taylor series corresponding to its integrand.
• Evaluating Limits by expressing all or part of the expression in terms of a series.

$f(x) = \int_0^x \frac{1}{1+t^5}\,dt$ can be thought of as approximating a function. We plugged in $x=.2$ in the 11/18/11 notes for 12.9.

We also built a power series for $\cos(x)$, centered at $a=0$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Using Taylor's Series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

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