12.9 REPRESENTATIONS OF FUNCTIONS AS POWER SERIES

Here we use some of the things we know from before about geometric series, in particular, to help us write all KINDS of functions as power series. We start with our old friend, the closed-form expression for the geometric series. Recall:

$$a + ar + ar^2 + ar^3 + \cdots = a\sum_{n=0}^{\infty} r^n = a\sum_{n=1}^{\infty} r^{n-1} = a\cdot\frac{1}{1-r}$$

$$\text{In particular, } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Turn it around, and *now* we look for a power series representation for objects like

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Now, there's NOTHING that says we can NOT simply do a change of variable, if we like, and every reason to hope that we CAN!

$$\text{Consider } \frac{1}{1-x^2}$$

If life were good and fair, we *would* be able to simply replace x by x^2 in the preceding... WE CAN!!!!

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots$$

$$= 1 + x^2 + x^4 + x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} x^{2n} \qquad !!!!!!$$

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Example 1 does something very similar with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

3–10 Find a power series representation for the function and determine the interval of convergence.

$$\frac{1}{1-x}$$

5. $f(x) = \dfrac{2}{3-x} = \dfrac{2}{3\left(1-\frac{x}{3}\right)} = \dfrac{2}{3} \cdot \dfrac{1}{1-\frac{x}{3}}$

$$= \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

$\boxed{8.}$ $f(x) = \dfrac{x^3}{2x^2+1} = \dfrac{x^3}{1-(-2x^2)} = x^3 \cdot \dfrac{1}{1-(-2x^2)}$

$= x^3 \sum_{n=0}^{\infty} (-2x^2)^n = x^3 \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot x^{2n}$

$= \sum_{n=0}^{\infty} (-1)^n 2^n \cdot x^{2n+3}$

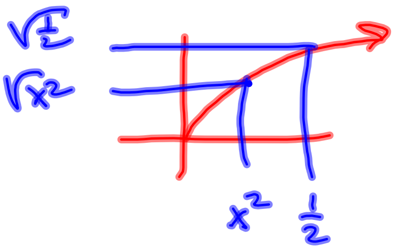
x^{2n+5}

Radius of convergence. $\left| \dfrac{(-1)^{n+1} \cdot 2^{n+1} \cdot x^{2(n+1)+3}}{(-1)^n \cdot 2^n \cdot x^{2n+3}} \right|$

$= |2 \cdot x^2| \overset{\text{Want}}{<} 1$

$\sqrt{x^2} < \sqrt{\frac{1}{2}}$

$\boxed{|x| < \frac{1}{\sqrt{2}}}$



Interval of convergence?

Check the endpoints of $\boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

$\sum_{n=0}^{\infty} (-1)^n (2)^n \left(-\frac{1}{\sqrt{2}}\right)^{2n+3} = \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \left(-1\right)^{2n+3} \left(2^{-\frac{1}{2}}\right)^{2n+3}$

$n - n - 3/2$

$n + 2n + 3$
$= 3n + 3$
$= 3(n+1)$

$= \sum_{n=0}^{\infty} (-1)^{3n+3} 2^n \left(2^{-n-\frac{3}{2}}\right)$

$= \sum_{n=0}^{\infty} (-1)^{n+1} 2^{-\frac{3}{2}}$

DIFFERENTIATION AND INTEGRATION OF POWER SERIES

The upshot of all this discussion is...

You may safely differentiate and integrate a convergent power series term-by-term, just as you might hope and expect you could! This FACT can be used in some unexpected, but very cool ways!

$\boxed{2}$ **THEOREM** If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

$f(a) = c_0$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

$f'(a) = c_1$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

But when you differentiate or integrate, term-by-term, you need to check the endpoints to know what the actual *interval* of convergence is. (Sometimes include. Sometimes not.

This has applications peeking around corners left and right. You'd be surprised.

Pattern recognition is going to be HUGE, here. But first, some notational tweaks that might be more along the lines of some practical applications:

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$$

(iv) $$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$$

Suppose $a=0$ & $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

Expansion about $a=0$.

$f(0) = c_0 = 0!c_0 \Longrightarrow c_0 = f(0) = \dfrac{f^{(0)}(0)}{0!}$

$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots$

$f'(0) = f^{(1)}(0) = c_1 = 1!c_1 \quad c_1 = f'(0) = \dfrac{f'(0)}{1!}$

$f''(x) = f^{(2)}(x) = 2c_2 + 3\cdot 2 c_3 x + 4\cdot 3 c_4 x^2 + 5\cdot 4 c_5 x^3 + \cdots$

$f^{(2)}(0) = 2c_2 = 2\cdot 1 c_2 = 2!c_2 \quad c_2 = \dfrac{f^{(2)}(0)}{2!}$

$f'''(x) = f^{(3)}(x) = 3\cdot 2 c_3 + 4\cdot 3\cdot 2 c_4 x + 5\cdot 4\cdot 3 c_5 x^2 + \cdots$

$f^{(3)}(0) = 3\cdot 2 c_3 = 3\cdot 2\cdot 1 c_3 = 3!c_3 \quad c_3 = \dfrac{f^{(3)}(0)}{3!}$

$f^{(4)}(x) = 4\cdot 3\cdot 2 c_4 + 5\cdot 4\cdot 3\cdot 2 c_5 x + 6\cdot 5\cdot 4\cdot 3 c_6 x^2$

$f^{(4)}(0) = 4\cdot 3\cdot 2 c_4 = 4!c_4 \quad c_4 = \dfrac{f^{(4)}(0)}{4!}$

$\vdots$

$f^{(n)}(0) = n!c_n \quad c_n = \dfrac{f^{(n)}(0)}{n!}$

Series Looks Like

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

Maclaurin's Series! $a=0$

Notice at $x=0$ that $c_0 = f(x)$ is exactly right.

FACT: The farther from $x=0$, the more terms we need to be accurate within a specified tolerance.

Taylor's Series - Expansion about $x=a$.

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

More accurate quicker when x is close to a.

More on this, later.

Maclaurin's series for $f(x)=\cos(x)$

$f(x)=\cos x \quad f(0)=1$

$f'(x)=-\sin x \quad f'(0)=0$

$f''(x)=-\cos x \quad f''(0)=-1$

$f'''(x)=\sin x \quad f'''(0)=0$

$f^{(4)}(x)=\cos x \quad f^{(4)}(0)=1$

$$\Rightarrow \cos(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 1+0\cdot x^1 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots$$

$$= 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\dots$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Radius of Convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2(n+1)}}{(2(n+1))!}}{\frac{x^{2n}}{(2n)!}}\right| = \left|\frac{x^{2n+2}}{(2n+2)(2n+1)(2n)!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= \left|\frac{x^2}{(2n+2)(2n+1)}\right| = \frac{1}{(2n+2)(2n+1)}x^2 \xrightarrow{n\to\infty} 0 \ \forall x$$

since it's ≥ 0 inside.

So Radius of convergence is ∞ and interval of convergence is $(-\infty,\infty)$

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There are a few interesting surprises these results give us. We can really expand our flight envelope quite a bit.

15–18 Find a power series representation for the function and determine the radius of convergence.

$\boxed{15.}$ $f(x) = \ln(5 - x)$

You gotta be kiddin' me!!! Ahhhh. We *do* know the derivative of this function! And its derivative has a nice power series representation. So to find what we're looking for, we first TAKE the derivative of f , find the power series for f, and then we take the term-by-term *antiderivative* to get the final answer.

And think of the possibilities for estimating integrals of some uglier functions that have warm 'n' fuzzy power series representations... You just integrate the power series term by term, and use as many terms as you need to achieve the desired accuracy.

More Spinoffs:

23–26 Evaluate the indefinite integral as a power series. What is the radius of convergence?

24. $\int \frac{\ln(1-t)}{t}\,dt$

27–30 Use a power series to approximate the definite integral to six decimal places.

See also: Example 8.

27. $\displaystyle\int_0^{0.2} \frac{1}{1+x^5}\,dx$

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Find a power series for

$$f(x) = \frac{7}{(1-x)^2}$$

$$f'(x) = -2\cdot 7(1-x)^{-1}(-1) = \frac{14}{1-x} = 14\cdot\frac{1}{1-x}$$

$$= 14\sum_{n=0}^{\infty} x^n \implies$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\boxed{f(x) = \left(14\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)} + C = \boxed{14\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} + 7}$$

and this will be given by $f(0)$.

$$f(0) = \frac{7}{1^2} = 7$$

$$f(x) = \frac{7x^5}{(1-x)^2} = x^5\left[14\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} + 7\right]$$

$$= 14\sum_{n=0}^{\infty}\frac{x^{n+6}}{n+1} + 7x^5$$

arctan(x) = f(x) Same deal

$f'(x) = \frac{1}{1+x^2}$, etc.

§12.9

See Example 7

f(x) = arctan(x)

$f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + f(0) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

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$$\boxed{\int_0^1 e^{x^2}dx} = \int_0^1 \sum_{n=0}^{7} \frac{f^{(n)}(.5)}{n!}(x-.5)^n\,dx$$

$e^{x^2} = f(x)$

$f'(x) = 2xe^{x^2}$

$f''(x) = 4x^3e^{x^2} + 2e^{x^2}$

$f'''(x) = 12x^2e^{x^2} + 4x^3 \cdot 2xe^{x^2}$

$f'(.5)$

$f''(.5)$

$f'''(.5)$