12.10 TAYLOR AND MACLAURIN SERIES

Assume that f can be represented by a power series on some interval centered at $x = a$. Then

1 $$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R$$

Where R is the radius of the interval in question. Note the interval described is an open interval. Typically, we'll need to check if the endpoints are included or not.

Here's the idea:

If f has such a representation, then $f(a)$ is given by

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \ldots = c_0$$

$$f(a) = c_0$$

Likewise, term by term differentiation reveals that

2 $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R$$

$$f'(a) = c_1$$

3 $$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2$$

$$\frac{f''(a)}{2} = \frac{f^{(2)}(a)}{2} = c_2$$

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$$

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

$$\frac{f'''(a)}{3 \cdot 2} = \frac{f^{(3)}(a)}{3!} = c_3$$

In General, then:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This gives us a nice algorithm for calculating the coefficients of a power series for *any* function that we can differentiate! COOL!

5 THEOREM If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Rewriting the power series expansion for f using this gives us:

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

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When the series is centered at $a = 0$, it's called the Maclaurin series or Maclaurin expansion for f. It's generally more convenient to use $a = 0$, especially when the radius of convergence is infinite.

Note for Future: But typically, we want to choose a fairly close to values of x at which we are likely to be evaluating f, because the closer x is to a, typically the faster the convergence of the series and the fewer terms we need in order to approximate f by a partial sum.

Maclaurin series

7 $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

The industry standard Maclaurin series example

V EXAMPLE 1 Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

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Pretty neat stuff. Just keep in mind, we've been *assuming* that f actually *has* a power series representation.

As our first requirement for the existence of such a representation, for this method to work, **clearly f must have derivatives of all orders**. (necessary, but not sufficient).

T_n is the n^{th} degree Taylor Polynomial. You'd like to think that as n grows large, that T_n gives a decent approximation for f. Follow the link to TEC, below for an illustration:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

In general, this representation "works" if T_n converges to f. So to see if such a thing is happening, we basically have to show that the remainder, R_n converges to zero in the following:

$$f(x) = \lim_{n\to\infty} T_n(x)$$

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

8 THEOREM If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

A tool for showing that R_n converges to zero (or for deciding how large n must be in order for T_n to be sufficiently close to f...

9 TAYLOR'S INEQUALITY If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leq d$$

So finding bounds on derivatives appears to be on the table.

A useful fact that we no longer have to justify every time we use it is the following:

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$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

EXAMPLE 4 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

The Maclaurin series is nice for $\sin x$, because the pattern is relatively easy to see and half the terms just vanish on their own. The Taylor series expansion for $\sin x$ doesn't generally have these terms vanish, but it sure is handy for the MacLaurin series that $\sin(0) = 0$ on all the even-index terms. The text does a nice writeup for one of these (and for cosine, too).

In fact, some of the examples the book worked out for you would've been nice exercises. It's almost too bad.

The Binomial Coefficient, also known as combinations of n things taken k at a time, is generalized quite a bit in this section, for those of you who've already been exposed to combinations, permutations, and the Binomial Theorem in previous courses.

This is discussed in Example 8.

17 THE BINOMIAL SERIES If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

What happens when k is a positive integer?

How would you BUILD the power series expansion for this?

An example that bootstraps off the binomial expansion, rather than just using the basic algorithm is given by:

EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION We write $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\dfrac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\dfrac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Now just apply the binomial expansion to the above. I never would've thought of this as a student.

A table of the standard Maclaurin series. Probably good material for a cheat sheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Applications of this theory include:

- Approximating a function by using its Taylor polynomial (Early calculators used Taylor polynomials to do 'most all their trig. calculations).
- Approximating a definite integral by integrating the first handful of terms of the Taylor series corresponding to its integrand.
- Evaluating Limits by expressing all or part of the expression in terms of a series.