12.8 Power Series

Each term is a power function.

1 $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

A power series whose coefficients are the same constant is just a geometric series, which converges for $-1 < x < 1$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

converges if $|r| < 1$

$$\sum_{n=1}^{\infty} cx^{n-1} = \boxed{\sum_{n=0}^{\infty} cx^n} = c + cx + cx^2 + cx^3 + \cdots = c\sum_{n=0}^{\infty} x^n = \boxed{\frac{c}{1-x}}$$

A power series in $x - a$ or "centered at $x = a$". You're basically shifting what you know about a power series in x either right ($a > 0$) or left ($a < 0$), the same as you shift $f(x)$ to the right 5 by replacing x by $x - 5$.

2 $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

The closer x is to a, the quicker it converges.

Heretofore, we had terms in which the only variable was the n in each term. Now, when we talk about convergence, we will want to know the values of x for which the series converges

For what values of x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent?

We know $\sum_{n=0}^{\infty} \frac{27^n}{n!}$ converges

Ratio Tests:

$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$\frac{n!}{(n+1)!} = \frac{\cancel{n\cdot(n-1)(n-2)\cdots}}{(n+1)\cancel{(n)(n-1)(n-2)\cdots}}$$

$$= |x| \frac{1}{n+1} \xrightarrow{n\to\infty} 0$$ so it converges!

Regardless of the value of x.

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

The language or parlance or lingo for dealing with power series is always in terms of convergence of partial sums, for instance,

$$J_0(x) = \lim_{n\to\infty} s_n(x)$$

Have a look at 12.8 Maple.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$\boxed{R = 1}$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$\boxed{R = 0}$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$\boxed{R = \infty}$	$\boxed{(-\infty, \infty)}$

R is the least upper bound on the distance x can be from the center, x = a, and still give a convergent series.

$\sum_{n=0}^{\infty} 27(x-5)^n$ How far from 5 can ya get?

$= 27 \sum_{n=0}^{\infty} (x-5)^n$ is geometric in (x-5).

Need $|x-5| < 1$, i.e.

$-1 < x-5 < 1$

$4 < x < 6$

$\boxed{R = 1}$ Interval of convergence is $(4, 6)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad a=1,\ r=x \qquad S = \frac{a}{1-x}$$

$$\boxed{\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} x^n + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(x^n + (-1)^n x^n\right)\right] = \frac{1}{2}\sum_{n=0}^{\infty}\left(x^n\left(1+(-1)^n\right)\right)$$

$$= \frac{1}{2}\left[1\left(1+(-1)^0\right) + x^1\left(1+(-1)^1\right) + x^2\left(1+(-1)^2\right) + \cdots\right]$$

$$= \frac{1}{2}\left[2 + x\cdot 0 + 2x^2 + 0x^3 + 2x^4 + 0x^5 + \cdots\right]$$

$$= \frac{1}{2}\cdot 2\left[1 + x^2 + x^4 + x^6 + \cdots\right]$$

$$= 1\sum_{n=0}^{\infty} x^{2n}$$

Beautiful

Find a power series representation for

$$\frac{1}{2x^2+1} = \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty} (-2x^2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-2)^n (x^2)^n} = \boxed{\sum_{n=0}^{\infty} (-1)^n 2^n x^{2n}}$$