12.2 #52

$$\sum \ln\left(1+\frac{1}{n}\right) = \sum \ln\left(\frac{n+1}{n}\right)$$

$$= \sum \left[\ln(n+1) - \ln(n)\right]$$

$$\sum_{k=1}^{4} \ln\left(1+\frac{1}{k}\right) = \ln(2) - \ln(1) + \ln(3) - \ln(2) + \ln(4) - \ln(3) + \ln(5) - \ln(4)$$

$$= -\ln(1) + \ln(5)$$

$$= 0 + \ln(5)$$

MIST / SITE
Lee McManus
John Mangin

$$\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \ln(n+1)$$

one bigger than n.

So it diverges, since the sequence of partial sums $\{S_n\} = \{\ln(n+1)\}$ increases without bound.

$\ln(x)$ increases without bound.

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x} > 0 \ \forall\ x \in \mathbb{R}.$$

$$\lim_{n\to\infty} \left(\ln\left(1+\frac{1}{n}\right)\right) = \ln(1) = 0$$

$\forall$

$\exists$

$\Longrightarrow$

S_{2^0} $S_1 = 1$

S_{2^1} $S_2 = 1 + \frac{1}{2}$

S_{2^2} $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$

S_{2^3} $S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= 1 + 1/2 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} + \underline{2}(\frac{1}{2})$

$S_{2^{\underline{4}}}$ $S_{16} = \cdots > 1 + \frac{1}{2} + \underline{3}(\frac{1}{2})$

$S_{2^n} > 1 + \frac{1}{2} + (n-1)\frac{1}{2}$ and we can make this bigger than any $M > 0$, by letting n get big.

Telescoping in general

$$\sum_{n=5}^{\infty} \frac{1}{n^2-4}$$

$1 = \boxed{A(n-2)} + B(n+2)$

$1 = -2A + 2B \qquad 0 = A + B$

$1 = -2(-B) + 2B$

$4B = 1$

$B = \frac{1}{4}$

$A = -\frac{1}{4}$

$$\sum_{n=5}^{\infty} \frac{1}{n^2-4} = \underbrace{\boxed{\sum_{n=5}^{3000000} \frac{1}{n^2-4}}}_{\text{FINITE}} + \underbrace{\boxed{\sum_{n=3000001}^{\infty} \frac{1}{n^2-4}}}_{\text{May be finite.}}$$

$$= \sum_{n=5}^{\infty} \left[\frac{1}{4} \cdot \frac{1}{n-2} + \frac{1}{4} \cdot \frac{1}{n+2}\right] = \frac{1}{4} \sum_{n=5}^{\infty} \left[\frac{1}{n-2} - \frac{1}{n+2}\right]$$

$$\frac{1}{4} \sum_{n=5}^{11} a_n = \frac{1}{3} - \frac{1}{7} + \frac{1}{4} - \frac{1}{8} + \frac{1}{5} - \frac{1}{9}$$
$$+ \frac{1}{6} - \frac{1}{10} + \frac{1}{7} - \frac{1}{11} + \frac{1}{8} - \frac{1}{12} + \frac{1}{9} - \frac{1}{13}$$

$$\frac{1}{n^2+4n+3}$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \underbrace{\frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \frac{1}{13}}$$

So you see that we always end up with $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \underbrace{\frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}}$

$$\frac{20+15+12+10}{60} = \frac{57}{60} = \frac{19}{20}$$

$$S_n = \sum_{k=1}^{n} a_n = \frac{57}{60} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} \xrightarrow{n \to \infty} \frac{57}{60} = \frac{19}{20}$$

$$\left\{\frac{19}{20} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right\}$$ is

the sequence of partial sums.

12.5 ALTERNATING SERIES

THE ALTERNATING SERIES TEST If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

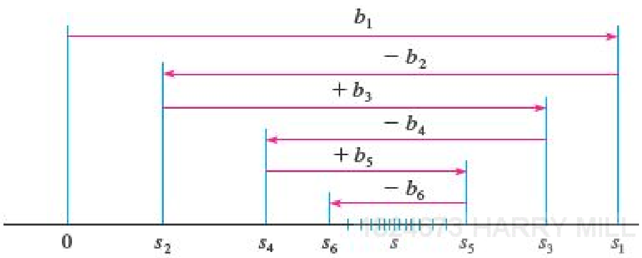

FIGURE 1

We don't always prove theorems, but this is a nice, easy argument that's good to see.

$$\frac{1}{1} - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} + \cdots\cdots$$

$a_n = \frac{1}{n}$ if $n = 2k-1$, $k = 1, 2, \ldots$

$a_n = \frac{1}{n^2}$ if $n = 2k$, $k = 1, 2, \ldots$

So $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges, right?

$b_{n+1} \le b_n$ $\qquad \sum (-1)^{n-1} b_n$

$b_n \xrightarrow{n \to \infty} 0$

$b_4 = \frac{1}{16} < b_5 = \frac{1}{5}$

It does NOT satisfy $b_{n+1} \le b_n$

Let $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

Then $\sum a_n$ Diverges, by p-test, basically.
and $\sum b_n$ Converges, "

and so $\sum (a_n - b_n)$ must diverge

$= 1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} + \cdots$

$$s_2 = b_1 - b_2 \geq 0 \qquad \text{since } b_2 \leq b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geq s_2 \qquad \text{since } b_4 \leq b_3$$

In general $\quad s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} \qquad \text{since } b_{2n} \leq b_{2n-1}$

Thus $\quad 0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \cdots$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

$$s_{2n} \leq b_1$$

Monotonic Sequence Theorem.

Even-index partial sums converge:

$$\lim_{n\to\infty} s_{2n} = s$$

Odd-index partial sums converge, since

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \\ &= s \end{aligned}$$

[see Exercise 80(a) in Section 12.1]

80. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Nov 10-11:04 AM

Notice that the Theorem does not give us a condition for divergence. So when (i) or (ii) aren't satisfied, we need to look closer. But we can probably do better than the Alternating Series Test, by saying that the series diverges if b_n does NOT approach zero as n approaches infinity (We already have that from our Test for Divergence), if (i) *is* satisfied, because, you know that the b_n's approach a limit by Monotone Sequence (Convergence) Theorem. If they approach a nonzero limit, then the alternation will cause 'em to bounce above and below zero enough to make the SUM bounce around too much.

2–20 Test the series for convergence or divergence.

4. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots$

Flunks the p-test, but because it alternates, it Does converge, CONDITIONALLY

But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ does NOT converge.

Alternating Harmonic Series Does converge:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES

§12.6 Absolute Convergence

$\sum b_n$ converges <u>absolutely</u> if

$\sum |b_n|$ converges.

Very <u>Strong</u> convergence.

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow{n\to\infty} L<1$$

Root Test

$$\sqrt[n]{|a_n|} \xrightarrow{n\to\infty} L<1$$

Converge Absolutely

$$\sum \frac{x^n}{n!}$$

Ratio and root tests are <u>Great</u> for factorials

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$
$$= (1)(2)(3)\cdots(n-2)(n-1)n$$

$$\sum \frac{2^n}{n!}$$

Ratio Test: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} =$

$$\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \frac{2\cdot 2^n\cdot n!}{(n+1)\cdot n!\cdot 2^n} = \frac{2}{n+1} \xrightarrow{n\to\infty} 0<1$$

converges

Divergent $\sum \frac{1}{n}$: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{\frac{1}{n+1}}{\frac{1}{n}} = \dfrac{1}{n+1}\cdot\dfrac{n}{1} = \dfrac{n}{n+1} \xrightarrow{n\to\infty} 1$

Test is inconclusive when $\left|\dfrac{a_{n+1}}{a_n}\right| \xrightarrow{n\to\infty} 1$

Convergent $\sum \frac{1}{n^2}$: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \dfrac{n^2}{(n+1)^2}$

$$= \frac{n^2}{n^2+2n+1} \xrightarrow{n\to\infty} 1$$ Inconclusive.

$$\sum \frac{n^n}{n!} = \sum \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}$$

Blows up

3 probs each

FOR TOMORROW

I want to see SOME work from 12.3, 12.4, 12.5, 12.6

$$2^n \qquad \sum \frac{27^n}{n!} \longrightarrow$$

$$1, x, x^2, \ldots, x^n < 2^x, 3^x, \ldots, n^x < n! < x^x$$

$$n^n$$

Nov 9-12:53 PM

#12 ought to compare favorably with an alternating harmonic series, but we might have some fun trying to SHOW that the b_n's converge monotonically to zero, although it may take a term or three, before monotone decrease kicks in (a detail you don't want to miss).

A nice way to get alternating signs is with an $n\pi$ inside a trig function:

16. $$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$$

(Our first factorial? $n!$ in the denominator grows pretty fast. How does it compare to exponential growth?)

Nov 10-11:08 AM

ALTERNATING SERIES ESTIMATION THEOREM If $s = \Sigma (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i) } 0 \leqslant b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii) } \lim_{n\to\infty} b_n = 0$$

then
$$|R_n| = |s - s_n| \leqslant b_{n+1}$$

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. It follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n| = b_{n+1} \qquad \square$$

21–22 Calculate the first 10 partial sums of the series and graph both the sequence of terms and the sequence of partial sums on the same screen. Estimate the error in using the 10th partial sum to approximate the total sum.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}}$ **22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3}}$

I did this a couple ways with technology:

1. With Excel and a pretty efficient method.
2. With Maple and an inefficient method.

I also used an inefficient method with Maple, and spent FAR too much time on it!!!

23–26 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \quad (|\text{error}| < 0.00005)$

$solve\left(\frac{1}{x^6} = .00005\right)$

5.210007310, 2.605003655 + 4.511998684 I, −2.605003655 + 4.511998684 I, −5.210007310, −2.605003655 − 4.511998684 I, 2.605003655 − 4.511998684 I

$g(6)$

$$\frac{-1}{46656}$$

$evalf(g(6))$

$$-0.00002143347051$$

$evalf(g(5))$

$$0.00006400000000$$

27–30 Approximate the sum of the series correct to four decimal places.

28. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$

See 12.5 Worksheet - By hand, you'd churn these out until you found a term that was less than .0001, and take the sum directly before.

32–34 For what values of p is each series convergent?

$\boxed{\textbf{32.}}$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

34. $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

I *think* we're just looking to see what it takes to make the sequence of terms to converge to zero.

35. Show that the series $\Sigma\,(-1)^{n-1}b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?

Nov 10-11:39 AM