How are you guys doing on Chapter 12, so far? 12.1 - 12.3? Is it going fairly quickly? Relative to my schedule, it seems that we're FLYING through the material, but it seems to me in lecture that the pacing is reasonable.

What this tells me is that either I'm reading things wrong, or *we have time to play with some of this stuff, and learn some older stuff BETTER !!!*

§12.1 #39

$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}$

$a_n = \left(1+\frac{2}{n}\right)^n$

$= \left(1+\frac{2}{n}\right)^{\frac{n}{2}\cdot 2}$

$= \left(\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right)^2$

$\xrightarrow{n\to\infty} e^2$

$A(t) = P\left(1+\frac{r}{m}\right)^{mt} = P\left(1+\frac{r}{m}\right)^{\frac{m}{r}\cdot rt}$

as $m \to \infty$,

$A(t) \longrightarrow Pe^{rt}$

$= P\left(\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right)^{rt}$

$\xrightarrow{m\to\infty} Pe^{rt}$

$\lim_{h\to 0} (1+h)^{\frac{1}{h}} = e$

See 418, 444

$\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^m = e$

r = rate

m = periods per year

$i = \frac{r}{m}$ = interest rate per period.

$I = Prt = Pr\cdot\frac{1}{m} = Pi$ = Interest per period.

Period	Balance
0	P
1	$P + Pi = P(1+i)$
2	$P(1+i) + P(1+i)i$ $= P(1+i)[1+i]$ $= P(1+i)^2$
3	.... $P(1+i)^3$

n periods: $P(1+i)^n$

Now, with t = yrs, $n = mt$

This gives $A = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$

As $m \to$ BIG, $A \longrightarrow Pe^{rt}$

Compounded Daily:

$$\boxed{P\left(1+\frac{r}{365}\right)^{365t}} \approx Pe^{rt}$$

Continuous Compounding.

$$\boxed{a_n = n\sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}} \xrightarrow{n\to\infty} \frac{0}{0}$$

$\infty \cdot 0$

$$\text{"}=\text{"}\ \frac{0}{\frac{1}{\infty}}\ \text{"}=\text{"}\ \frac{0}{0}$$

L'Hôpital handles this

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

12.4 THE COMPARISON TESTS

THE COMPARISON TEST Suppose that Σa_n and Σb_n are series with positive terms.

(i) If Σb_n is convergent and $a_n \leq b_n$ for all n, then Σa_n is also convergent.

(ii) If Σb_n is divergent and $a_n \geq b_n$ for all n, then Σa_n is also divergent.

So, if it's term-by-term *less* than a convergent series, it converges.
If it's term-by-term *greater* than a *divergent* series, it *diverges*.

Give examples and, per **Note 2**, in the text, a non-example or two, to show the limitations of the (direct) comparison test.

(ii) $\sum \frac{1}{\sqrt{n}}$ diverges, because $\frac{1}{\sqrt{n}} > \frac{1}{n}$ & we know the harmonic series $\sum \frac{1}{n}$ diverges.

(i) $\sum \frac{1}{n^2+1}$ converges, because $\sum \frac{1}{n^2}$ converges, and $\frac{1}{n^2+1} < \frac{1}{n^2}$ $\forall n \in \mathbb{N}$.

Be nice if $\sum \frac{1}{n^2-1}$ converged. But DIRECT comparison to $\sum \frac{1}{n^2}$ doesn't work.

$\frac{1}{n^2-1} > \frac{1}{n^2}$. This is where the LIMIT COMPARISON test comes in.

$a_n = \frac{1}{n^2-1}$ should remind you of $\frac{1}{n^2} = b_n$ should be thinking convergent, due to p-test.

$$\frac{a_n}{b_n} = \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \frac{1}{n^2-1} \cdot \frac{n^2}{1} = \frac{n^2}{n^2-1} \xrightarrow{n\to\infty} 1 > 0$$

Since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{1}{n^2-1}$

$$\sum_{n=100}^{\infty} \frac{n^2+n+1}{\sqrt[3]{n^7-n^4-11}} \qquad \frac{n^2}{\sqrt{n^7}} = \frac{n^2}{n^{7/2}} = \frac{1}{n^{3/2}} = b_n$$

$$\frac{a_n}{b_n} = \frac{\frac{n^2+n+1}{\sqrt[3]{n^7-n^4-11}}}{\frac{1}{n^{3/2}}} = \frac{n^2+n+1}{\sqrt[3]{n^7-n^4-11}} \cdot \frac{n^{\frac{3}{2}}}{1} \qquad \frac{3}{2}+\frac{4}{2}=\frac{7}{2}$$

$$= \frac{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{\sqrt[3]{n^7(1-n^{-3}-11n^{-7})}} \cdot \frac{n^{\frac{3}{2}}}{1} = \frac{n^{7/2}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{n^{7/2}\sqrt[3]{1-n^{-3}-11n^{-7}}}$$

$$= \frac{1+\frac{1}{n}+\frac{1}{n^2}}{\sqrt[3]{1-\frac{1}{n^3}-\frac{11}{n^7}}} \xrightarrow{n\to\infty} \frac{1}{3}.$$

So $\sum a_n$ converges, because $\sum b_n$ does.

An Example from the Limit Comparison Test, in the sequel, but a clever student might be able to find a way to apply the Comparison Test, directly.

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converges or diverges.

Tricky: $\frac{2n^2+3n}{\sqrt{5+n^5}} < \frac{2n^2+3n}{\sqrt{n^5}} < \frac{5n^2}{\sqrt{n^5}}$ for $n>1$

This suggests it diverges.

$= \frac{5n^2}{n^{5/2}} = \frac{5}{n^{1/2}}$ Not going to converge,

So no help, since these estimates are topsy-turvy. I was making the kinds of estimates one makes in order to show convergent.

If I can show it's BIGGER than something divergent, I'm done.

$$\frac{2n^2+3n}{\sqrt{5+n^5}} > \frac{2n^2}{\sqrt{5+n^5}} > \frac{2n^2}{\sqrt{n^5+n^5}} \quad \text{for } n>1$$

$$= \frac{2n^2}{\sqrt{2n^5}} = \frac{2n^2}{\sqrt{2}\, n^{5/2}} = \frac{2}{\sqrt{2}} \cdot \frac{1}{n^{1/2}}$$

So, it's bigger term-by-term $(n>1)$ than $b_n = \frac{2}{\sqrt{2}\, n^{1/2}}$ & we know $\sum \frac{2}{\sqrt{2}\, n^{1/2}}$ diverges.

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

compare (in the limit) to $\sum \frac{1}{\sqrt{n}} = b_n$

$$\frac{a_n}{b_n} = \frac{2n^2+3n}{\sqrt{n^5+5}} \cdot \frac{n^{\frac{1}{2}}}{1} = \frac{2n^{5/2}+3n^{3/2}}{n^{5/2}\sqrt{1+\frac{5}{n^5}}} \xrightarrow{n\to\infty} 2$$

$\sqrt{1+\frac{5}{n^5}} \to 1$

$n^{5/2}\left(2+\frac{3}{n}\right)$

§ 12.2 #52

$\sum \frac{1}{n}$

Show that $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ diverges.

clearly $\left\{\ln\left(1+\frac{1}{n}\right)\right\}$ is a decreasing sequence

$$\ln\left(1+\frac{1}{n}\right) - \frac{1}{n} = \frac{n\ln\left(1+\frac{1}{n}\right) - 1}{n}$$

$$\frac{x\ln\left(1+\frac{1}{x}\right) - 1}{x}$$

$$y = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}$$

$$y' = \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} - \frac{1}{x^2} < 0$$

$$= -\frac{1}{x^2\left(x+\frac{1}{x}\right)} - \frac{1}{x^2}$$

$$\ln\left(1+\frac{1}{1}\right) - \frac{1}{1}$$
$$= \ln(2) - 1 < 0$$

$\boxed{\text{Come back to it,}}$ Mañana.

So, $\ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$ @ $x = 1$, and remains so.

NOTE 1 Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

$\sum \frac{\ln(n)}{n}$

EXAMPLE 2 Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence.

Text uses the term "ultimately." Other terms for this include "eventually" or "for all but finitely many."

The point is that these comparison tests are built for series whose terms are positive and decreasing.

Note: $a_1 = \frac{\ln(1)}{1} = 0$

$a_2 = \frac{\ln(2)}{2} > 0$

So, even though a_1 doesn't fit the hypotheses, it does work for $a_2, a_3, \ldots$

By "eventual" comparison to $\sum \frac{1}{n}$, we know that $\sum \frac{\ln(n)}{n}$ diverges, since, for $n \geq 3$, $\ln(n) > 1$, and so $\frac{\ln(n)}{n} > \frac{1}{n}$ for $n \geq 3$.

For what values of p does

#38 $\sum \frac{1}{n^p \ln(n)}$ converge?

Integral Test Sucked $\int_1^\infty \frac{dx}{x^p \ln(x)}$ Owie!

Look at cases: $p = 0$:

$$\sum \frac{1}{\ln(n)} \not\to \qquad \frac{1}{\ln(n)} > \frac{1}{n}$$

$$p < 0$$

$$\sum \frac{1}{n^p \ln(n)} = \sum \frac{n^{-p}}{\ln(n)} \not\to \qquad n^{-p} \to >1$$

You look @ $0 < p < 1$, $p = 1$, $p > 1$

§12.2 #70

$\sum a_n$ & $\sum b_n$ both diverge. Does

$\sum (a_n+b_n)$ necessarily diverge?

No $\quad a_n = -1$, $\quad b_n = +1$

$$\sum(a_n+b_n) = \sum 0 = 0$$

If $\sum a_n$ converges & $\sum b_n$ diverge, does

$\sum (a_n+b_n)$ converge? No.

$\boxed{\text{Proof}}$ Suppose it DOES converge.

Then $\sum(a_n+b_n)$ & $\sum a_n$ converge.

By Theorem, if they both converge, so does their term-by-term sum (difference):

$$= \underbrace{\sum(a_n+b_n)}_{\text{must converge}}^{\to L} - \sum a_n^{\to M} = \sum(a_n+b_n-a_n) = \sum b_n$$

$\times$. $\blacksquare$

Then $\sum(a_n+b_n) - \sum a_n = L - M = \sum b_n$

$\times$

The following test will handle the non-examples I provided in the straight-ahead Comparison Test. It'll also clobber anything the Comparison Test will clobber. But the Comparison Test is quicker and cleaner when it *does* work.

THE LIMIT COMPARISON TEST Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

3–32 Determine whether the series converges or diverges.

5. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

6. $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

12. $$\sum_{n=0}^{\infty} \frac{1 + \sin n}{10^n}$$

Nov 4-10:29 AM

ESTIMATING SUMS

The idea is that you can use the "nicer" series, that you used to compare to your "ugly" series to get a handle on the error in the first n terms of your ugly series.

If you used the comparison test on $s = \sum_{k=1}^{\infty} a_k$,

comparing it to $t = \sum_{k=1}^{\infty} b_k$, which converges, *and has all positive terms* (or eventually does...), then from the fact that $a_n \le b_n$, we find that $R_n \le T_n$, where

$$R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k \text{ and } T_n = t - t_n = \sum_{k=n+1}^{\infty} b_k .$$

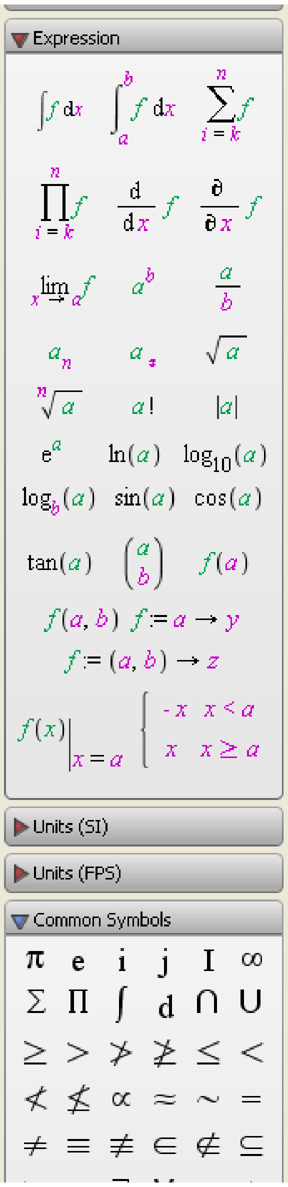
Since you usually pick a series that's EASIER to compare with, usually the integration estimates we make are EASIER to make for the T_n.

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

36. $\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$

	A	B	C
1	k	a_k	s_k
2	1	0.166666667	0.166666667
3	2	0.074074074	0.240740741
4	3	0.027777778	0.268518519
5	4	0.009876543	0.278395062
6	5	0.003429355	0.281824417
7	6	0.001175779	0.283000196
8	7	0.000400091	0.283400287
9	8	0.000135481	0.283535768
10	9	4.57247E-05	0.283581493
11	10	1.53955E-05	0.283596888
12	11	5.17461E-06	0.283602063
13	12	1.73693E-06	0.2836038
14	13	5.82424E-07	0.283604382
15	14	1.95137E-07	0.283604577
16	15	6.5336E-08	0.283604643
17	16	2.18641E-08	0.283604665
18	17	7.31333E-09	0.283604672
19	18	2.44532E-09	0.283604674
20	19	8.17372E-10	0.283604675
21	20	2.7314E-10	0.283604676
22	21	9.12537E-11	0.283604676
23	22	3.04809E-11	0.283604676
24	23	1.01795E-11	0.283604676
25	24	3.39908E-12	0.283604676
26	25	1.13484E-12	0.283604676
27	26	3.78841E-13	0.283604676
28	27	1.26454E-13	0.283604676
29	28	4.22051E-14	0.283604676

Nov 4-10:29 AM



$$\int_{10}^{\infty} \frac{x}{(x+1)\cdot 3^{x}}\, dx$$

$$-\frac{1}{59049}\,\frac{177147\,\mathrm{Ei}(1, 11\ln(3))\ln(3)-1}{\ln(3)}$$

$evalf(\%)$

$$0.00001411402009$$

$$\int_{10}^{\infty} \frac{1}{3^{x}}\, dx$$

$$\frac{1}{59049\ln(3)}$$

$evalf(\%)$

$$0.00001541498123$$

This one is more for *my* fun. At worst, it'll be a bonus question on the next test. Speaking of which, how do we want to handle this? We're getting ahead of schedule, here. Might want to stop and do some reviewing before we plunge too deeply into the later material in Chapter 12.

One issue students have with Chapter 12 is that the different tests, especially the Ratio Test -versus- Limit Comparison Test, get confused in their minds... I'd like to try to structure things to avoid that, as best I might.

39. Prove that if $a_n \geqslant 0$ and $\Sigma\, a_n$ converges, then $\Sigma\, a_n^2$ also converges.