§12.2 Series — Sums of sequences.

$a_1, a_2, \ldots, a_n, \ldots$ is a sequence

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is the corresponding series.

Notation $\sum_{n=1}^{\infty} a_n = \sum a_n$ is OK, as long as context is clear.

Given $S = \sum_{n=1}^{\infty} a_n$,

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is the n^{th} partial sum. We say that $\sum_{n=1}^{\infty} a_n$ converges if the sequence of n^{th} partial sums $\{S_n\}$ converges, and we write $\sum_{n=1}^{\infty} a_n = L$ if $S_n \xrightarrow{n\to\infty} L$.

$\sum_{K=1}^{\infty} \frac{1}{K^2}$ converges, by p-test for INTEGRALS!

$\int_1^{\infty} \frac{1}{x^2}\,dx$ Improper Integrals provide insight into convergence properties of series.

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

$n=1$ $\quad n=2$

$$= \sum_{n=1}^{\infty} ar^{n-1}$$

we find the sum.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$- \quad rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a \qquad - ar^n$$

$$S_n(1-r) = a - ar^n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} \xrightarrow{n\to\infty} \frac{a}{1-r}, \text{ if } |r|<1$$

Converges if $|r|<1$

$r \geq 1$ Bad

$r \leq -1$ Bad

$|r|<1$

$$\left(\tfrac{1}{2}\right)^n \xrightarrow{n\to\infty} 0$$

$$\left(\tfrac{999}{1000}\right)^n \xrightarrow{n\to\infty} 0$$

$S_1 = a$, $S_2 = a + ar$, $S_3 = a + ar + ar^2$

Look at the sequence $\{S_n\}$

vanishes if $|r|<1$

§ 12.2 I #s 1, 2, 3, 7, 9 - 12, 16, 19, 20, 22, 25, 29-31

12.2 II #s 35, 36, 41, 42, 47, 48, 52, 65, 69, 70

See TB discussion.

$0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1}$

$= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots$

$= 1$

But the geometric series w/ $r = -1$ diverges, so to say it equals zero to begin your discussion is Ludakriss.

$\sum_{k=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots$

Depending on how you pair them, you can get $+1$, 0 out of this.

$S'_1 = 1$

$S'_2 = 0$

$S'_3 = 1$

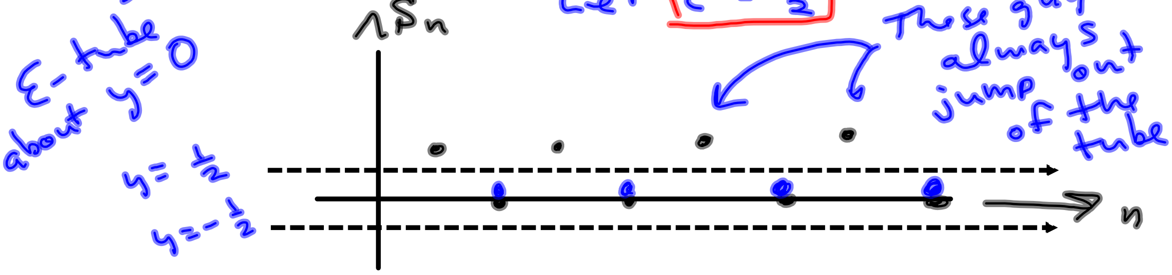
$S'_4 = 0$

$S'_5 = 1$

This gives $\{S'_n\} = \{1, 0, 1, 0, 1, \ldots\}$

Does NOT Converge.

Let $\boxed{\varepsilon = \frac{1}{2}}$



Ditch #8

Consider $\{a_n\} = \left\{\frac{n^2}{11n^2-1}\right\}$

(a) Does $\{a_n\}$ converge? If so, to what?

$a_n \xrightarrow{n \to \infty} \frac{1}{11}$

(b) $\boxed{\text{Does } \sum_{k=1}^{\infty} a_k}$ converge?

$\infty \cdot \frac{1}{11} = \infty$

$S_n = \sum_{k=1}^{n} \frac{k^2}{11k^2-1}$

$\approx a_1 + a_2 + \cdots + a_{5000000} + \frac{1}{11} + \frac{1}{11} + \frac{1}{11} + \frac{1}{11} + \frac{1}{11} + \frac{1}{11} \cdots$

It is NECESSARY (but <u>not sufficient</u>) for the terms (a_k's) to converge to zero.

Does $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ converge?

No. The terms are <u>growing</u> in absolute value

kinda the picture

How 'bout $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{e^{n-1} \cdot e}{3^{n-1}} = \sum_{n=1}^{\infty} e \cdot \frac{e^{n-1}}{3^{n-1}}$

$3^{n-1} = 3^n \cdot 3^{-1}$

$e^n = e^{n-1+1} = e^{n-1} \cdot e$

$= \sum_{n=1}^{\infty} e\left(\frac{e}{3}\right)^{n-1}$

$= \sum_{n=1}^{\infty} a r^{n-1}$

converges because

$r = \frac{e}{3} < 1$

$\sum_{k=1}^{\infty} \frac{1}{k}$ is harmonic series & it does **NOT** converge.

Flunks the integral test: $\int_1^{\infty} \frac{dx}{x}$ $\nexists$

So you see that $a_n \to 0$ is **NOT** sufficient.

$\frac{5}{6} - \frac{1}{4} = \frac{10-3}{12}$

$\frac{7}{12} + \frac{1}{5} = \frac{47}{60}$

$\sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\cdots$

$1, \frac{1}{2}, \frac{5}{6}, \frac{7}{12}, \frac{47}{60}$

Converges!

For an ALTERNATING series, $a_n \to 0$ is necessary AND sufficient.

Recall: If $\sum a_n$ & $\sum b_n$ converge to A & B, respectively, then $\sum(a_n+b_n)$ converges to $A+B$, so

$$\boxed{\sum(a_n+b_n)} = \boxed{\sum a_n} + \boxed{\sum b_n}$$ in this case.

$\sum 1$, $-\sum 1$ Both diverge, even though

$\sum(1-1) = \sum 0 = 0$ converges

If $\sum(a_n+b_n)$ converges, but $\sum a_n$ diverges, does $\sum b_n$ converge?

$$\sum \frac{1+2^n}{3^n} = \sum\left(\frac{1}{3^n}+\frac{2^n}{3^n}\right) = \sum\left(\frac{1}{3^n}+\left(\frac{2}{3}\right)^n\right)$$

converges, because $\sum \frac{1}{3^n}$ & $\sum\left(\frac{2}{3}\right)^n$ both converge.

We find the sum:

$$\sum_{n=1}^{\infty}\frac{1}{3^n} = \sum_{n=1}^{\infty} 1\cdot\left(\frac{1}{3}\right)^n = \frac{1}{1-\left(\frac{1}{3}\right)} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

New p.

Geometric Series $\displaystyle\sum_{k=1}^{\infty} a\cdot r^{k-1} = \frac{a}{1-r}$ if $|r|<1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^1\left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1-\left(\frac{1}{3}\right)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty}\frac{2}{3}\cdot\left(\frac{2}{3}\right)^{n-1} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = \frac{2}{3}\cdot\frac{3}{1} = 2$$

$$\therefore \sum_{n=1}^{\infty}\frac{1+2^n}{3^n} = \frac{1}{2}+2 = \boxed{\frac{5}{2}}$$

Totally different!

$\sum_{k=1}^{2} a_k \qquad \sum_{k=1}^{2} b_k$

$\left(\sum a_k\right)\left(\sum b_k\right) = (a_1+a_2)(b_1+b_2)$

$= a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$

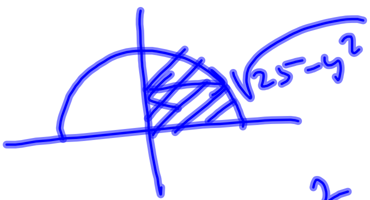
$\sum_{k=1}^{2} (a_k b_k) = a_1 b_1 + a_2 b_2$

$\int_0^5 (7-y)\sqrt{25-y^2}\,dy$

$= \int_0^5 7\sqrt{25-y^2} - \int_0^5 y\sqrt{25-y^2}\,dy$

$u = 25-y^2$

$du = -2y\,dy$



$7\cdot\frac{\pi\cdot 5^2}{4}$

$+\frac{1}{2}\int_0^5 \sqrt{25-y^2}\,(-2y\,dy)$

$= \frac{1}{2}(25-y^2)^{3/2}\cdot\frac{2}{3}\Big]_0^5$

$$\sum_{n=1}^{\infty} \ln\left(\frac{3n-1}{5n+2}\right) \not\to$$

$$\lim_{n\to\infty} \ln\left(\frac{3n-1}{5n+2}\right) = \ln\left(\lim_{n\to\infty}\left(\frac{3n-1}{5n+2}\right)\right) = \ln\left(\frac{3}{5}\right) \neq 0$$

$$\not\to$$

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) \not\to$$

$$\lim_{n\to\infty}\left(\ln\left(1+\frac{1}{n}\right)\right) = 0$$ is necessary but <u>not</u> sufficient.

$$\int_1^{\infty} \ln\left(1+\frac{1}{x}\right)dx = \infty$$

(i.e., diverges)

see if <u>this</u> converges. If it does, then so does $\sum \ln(1+\frac{1}{n})$

$\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ is toughie, but partial fractions!

You could use a comparison to $\frac{2}{n^2}$, only $\frac{2}{n^2-1} > \frac{2}{n^2}$

Intuition! p-test $p=2>1$ Yes. Converges.

Comparison to $\frac{2}{n^2+1}$ would compare to $\frac{2}{n^2}$, because $\frac{2}{n^2+1} < \frac{2}{n^2}$

$$\frac{2}{n^2-1} = \frac{A}{n+1} + \frac{B}{n-1}$$

$$2 = A(n-1) + B(n+1) = An - A + Bn + B$$

$$B - A = 2 \Rightarrow B = A + 2$$

$$A + B = 0 \Rightarrow A + B = A + A + 2 = 0 \Rightarrow 2A = -2 \Rightarrow A = -1$$

$$\Rightarrow B = 1$$

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \cdots$$

$n=6$

From this, we see that

$$S_n = \sum_{k=2}^{n} \frac{1}{k^2-1} = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

is a nice, closed-form expression for the n^{th} partial sum.

So $\{S_n\} = \left\{\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right\}$

$$S_n \xrightarrow{n \to \infty} \frac{3}{2} = \sum_{k=1}^{\infty} \frac{1}{k^2-1}$$

Rarely do we find this so easily.

§12.1 Spares

Increasing sequence $a_1 < a_2 < a_3 < \cdots$
Decreasing " $a_1 > a_2 > a_3 > \cdots$ } Monotone

Bounded Sequence:
$\{a_n\}$ is bounded above then $\exists M \in \mathbb{R} \ni$
$a_n \le M$ for every $n \in \mathbb{N}$

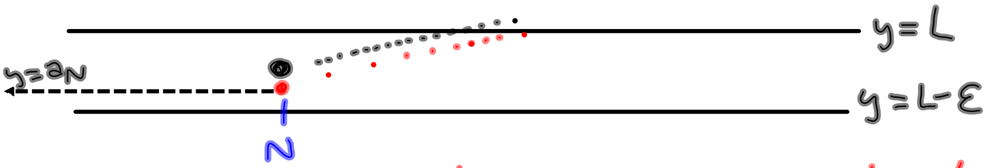
$\{\frac{1}{n}\}$ is bdd above by $M=2$ (or $M=1$)
$M=1$ is the LEAST upper bound.

Completeness Axiom – Every bounded set of real numbers has a LEAST UPPER BOUND and a GREATEST LOWER BOUND.

$\boxed{T12}$ MONOTONE CONVERGENCE THEOREM

A bounded monotone sequence converges.

Proof for bdd above, monotone increasing:



Suppose $\{a_n\}$ is a monotone increasing sequence, bounded above. Let L = the LEAST upper bound. Let $\varepsilon > 0$ be given. Then $L-\varepsilon$ is NOT an upper bound. Then $\exists N \in \mathbb{N} \ni a_N > L-\varepsilon$. $\{a_n\}$ is increasing says $a_n > L-\varepsilon \;\; \forall n \ge N$. So

$$L - a_n < \varepsilon \quad \forall n > N.$$

$$|L - a_n| < \varepsilon \quad \forall n > N. \blacksquare$$

$$\left(\lim_{n\to\infty} a_n = L\right.$$

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If $\sum_{k=1}^{\infty} a_k$ converges, then

$\sum_{k=1}^{N} a_k$ is an estimate for $\sum_{k=1}^{\infty} a_k$

The error, or remainder, R_N, is

$$R_N = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{N} a_k = \underbrace{\sum_{k=N+1}^{\infty} a_k}_{\text{The N-tail.}}$$

And if it converges,

then $\lim_{n\to\infty} R_n = 0$

Next time: For a monotone, decreasing, positive sequence $\{a_k\}$, we have the following estimate

$$\cancel{R_N = \sum_{k=N+1}^{\infty} a_k \leq \int_N^{\infty} f(x)\,dx \leq \sum_{k=N}^{\infty} a_k}$$

$$R_N \leq \int \leq R_{N-1}$$