§12.1 Sequences

$\boxed{D}$ A sequence is a list of #s written in a particular order.

$a_1, a_2, a_3, \ldots$

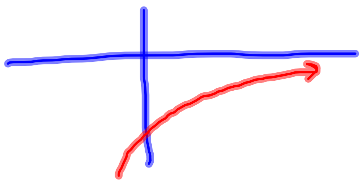
$\boxed{\text{Alternate}}$ A sequence is a function from $\mathbb{N}$ into $\mathbb{R}$, i.e., $f: \mathbb{N} \longrightarrow \mathbb{R}$

$a_1 = f(1), a_2 = f(2), \ldots$

This alternate lets us use what we know about functions from $\mathbb{R}$ into $\mathbb{R}$ to say things about sequences. $f: \mathbb{R} \longrightarrow \mathbb{R}$

$\boxed{\text{Notation}}$ $\{a_1, a_2, a_3, \ldots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$

$\boxed{\text{Examples}}$ $\left\{\frac{\ln(n)}{n}\right\}$



Application of "Alternate"

Fact: $\frac{\ln(n)}{n}$ converges to 0.

$$\lim_{x \to \infty} \frac{\ln x}{x} \overset{\text{L'HOP}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

$\{n\} = \{1, 2, 3, \ldots\}$

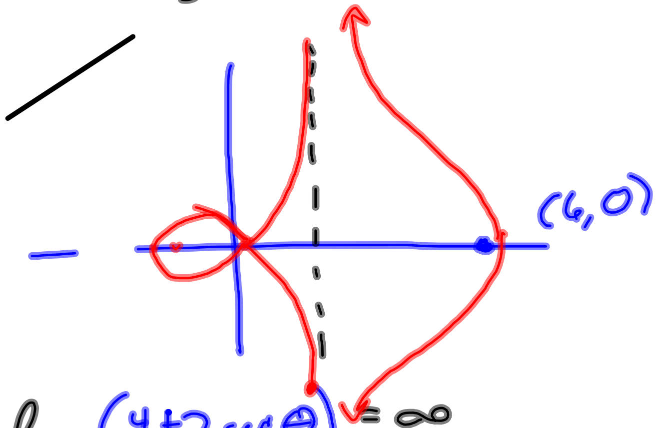
$\left\{(-1)^n \cdot \frac{1}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$

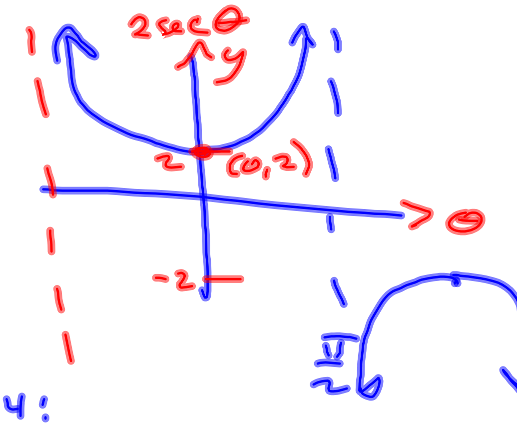
$\left\{(-1)^{n+1} \cdot \frac{1}{n}\right\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right\} = \left\{(-1)^{n-1} \cdot \frac{1}{n}\right\}$

§11.3 #51

$r = 4 + 2\sec\theta$

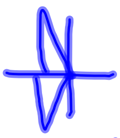
Show that it approaches $x = 2$ as $r \to \pm\infty$.

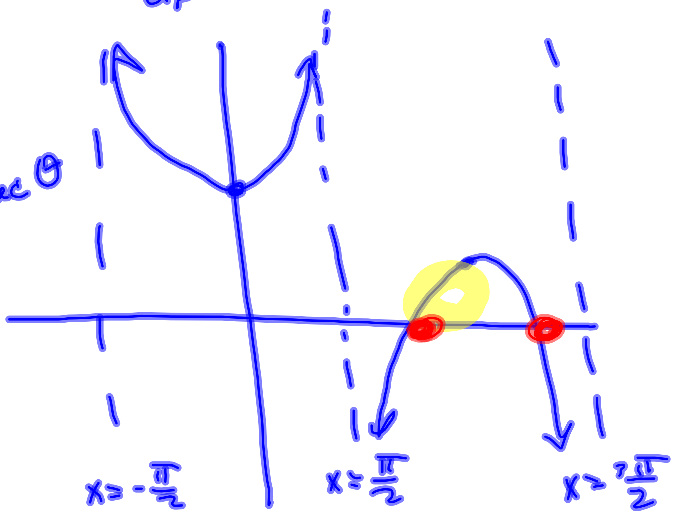




$$\lim_{\theta\to\frac{\pi}{2}^-} (4 + 2\sec\theta) = \infty$$

up 4!





$x^2 + y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

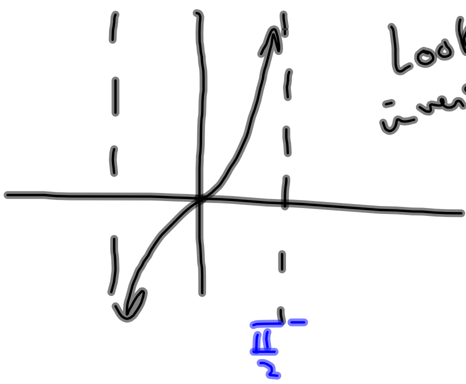
$$x = (4 + 2\sec\theta)\cos\theta$$

$$= 4\cos\theta + 2 \xrightarrow{\theta\to\frac{\pi}{2}^-} 2$$

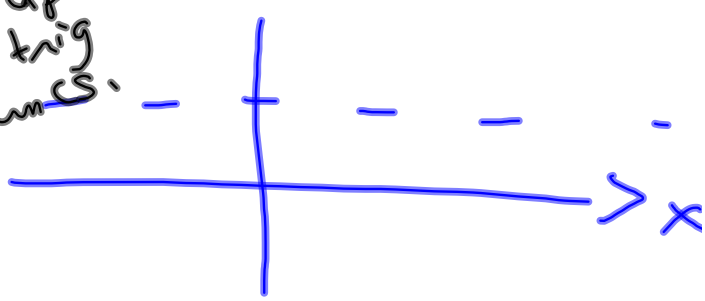
$$y = r\sin\theta$$

$$= (4 + 2\sec\theta)\sin\theta$$

$$= 4\sin\theta + 2\tan\theta \xrightarrow{\theta\to\frac{\pi}{2}^-} 4 + 2\lim_{\theta\to\frac{\pi}{2}^-} \tan\theta = +\infty$$



Look up inverse trig funcs.



Fun Puzzles Find a_n in general:

#10 $\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\}$

$a_n = f(n) =$ the # f assigns to the natural # n

$a_1 = 1$

$a_2 = \frac{1}{3} = \frac{1}{3} \cdot a_1$ Powers of $\frac{1}{3}$

2 is the input
$(\frac{1}{3})^1$ is output

$a_3 = \frac{1}{9}$

3 is input
$(\frac{1}{3})^2$ is output

$\boxed{a_n = (\frac{1}{3})^{n-1}}$

#13 $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{+16}{81}$ ($n=1$, $n=2$, $n=3$, $n=4$)

Alternating Signs

Look for $(-1)^n$ or $(-1)^{n+1}$ or $(-1)^{n-1}$

$(-1)^{n-1}$ handles signs.

-1

1,

$n=2$: $\frac{2}{3} = \frac{n}{n+1}$, $n=3$: $\frac{4}{9} = \frac{n}{n+1}$ Nope! Nope!

$n=2$: $\frac{2}{3}$ $n=3$: $\frac{4}{9} = \frac{2 \cdot 2}{3 \cdot 3} = \frac{2^2}{3^2} = \frac{2^{n-1}}{3^{n-1}} = \left(\frac{2}{3}\right)^{n-1}$

$(-1)^{n-1} \cdot \left(\frac{2}{3}\right)^{n-1}$, $n = 1, 2, 3, \dots$

$n=1$
$(-1)^{1-1} \left(\frac{2}{3}\right)^{1-1} = 1 \cdot 1 = 1$

$n=2$
$(-1)^{2-1} \left(\frac{2}{3}\right)^{2-1} = (-1)\,\frac{2}{3} = -\frac{2}{3}$ ✓

$2\,\overline{)6}$
3

$2\,\overline{)24}$
$2\,\overline{)12}$
$2\,\overline{)6}$
3

$1, \; x, \; \frac{x^2}{2}, \; \frac{x^3}{6}, \; \frac{x^4}{24}, \; \frac{x^5}{120}$

Try $n=0$ $n=1$

$24 = 2 \cdot 3 \cdot 4$

$120 = 2 \cdot 3 \cdot 4 \cdot 5$

$e_n = \frac{x^n}{n!}$

$n=0 \quad \frac{x^0}{0!} = 1$

$n=1 \quad \frac{x^1}{1!} = x$

$n=2 \quad \frac{x^2}{2!} = \frac{x^2}{2}$

$n=3 \quad \frac{x^3}{3!} = \frac{x^3}{6}$

$\boxed{D}$ $\{a_n\}$ has limit L and we write $\{a_n\}$

$\lim\limits_{n\to\infty} a_n = L$ OR $a_n \to L$ as $n \to \infty$ $\left\{\frac{x^n}{n!}\right\}$

OR $a_n \xrightarrow{n\to\infty} L$

Means $\left\{\frac{1}{n}\right\}$ $\frac{1}{n} \xrightarrow{n\to\infty} 0$

if we can make a_n as close to L as we like by taking n sufficiently large.

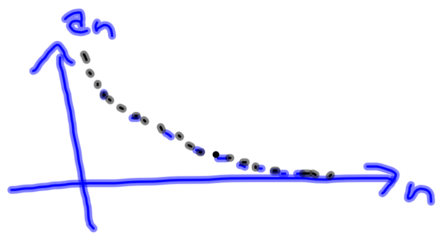
(And every a_n after that will also be within the desired tolerance.)

If $a_n \xrightarrow{n\to\infty} L$, $\{a_n\}$ converges.

(N) diverges.

$\boxed{E}$ Harmonic Sequence

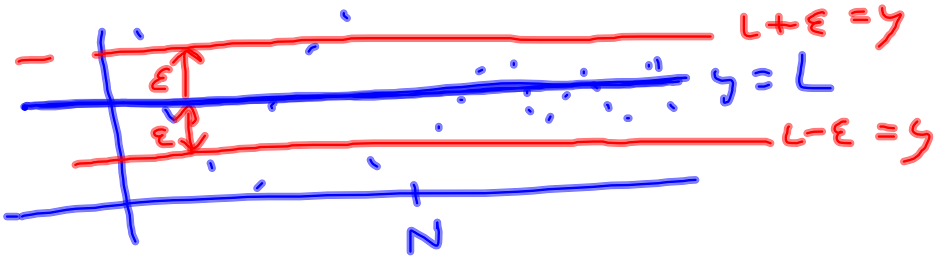
$\left\{\frac{1}{n}\right\}$. $\frac{1}{n} \xrightarrow{n\to\infty} 0$



$\boxed{D2}$ $\{a_n\}$ has limit L, i.e., $a_n \xrightarrow{n\to\infty} L$

iff $\Leftrightarrow$

$\forall \varepsilon > 0, \exists N \in \mathbb{N} \ni$ if $n > N$ then $|a_n - L| < \varepsilon$.



$(a_n \xrightarrow{n\to\infty} L)$ iff ($\forall \varepsilon>0\ \exists N\in\mathbb{N} \ni n>N$
$\Longrightarrow |a_n - L| < \varepsilon$)

$\Longleftrightarrow$

<u>Claim</u> $\lim_{n\to\infty} \frac{1}{n} = 0 = L$

Scratch:
want
$\frac{1}{n} < \varepsilon \Longrightarrow$
$1 < \varepsilon n \Longrightarrow$
$N \equiv \frac{1}{\varepsilon} < n$

<u>Proof</u> Let $\varepsilon > 0$. Define $N = \frac{1}{\varepsilon}$. Then if $n > N = \frac{1}{\varepsilon}$, we have $n > \frac{1}{\varepsilon} \Longrightarrow$

$$\boxed{\varepsilon > \frac{1}{n}} \quad \blacksquare$$

for an

Just like error estimates in Chapter 8 for trapezoid method.

#15 1st 6 terms of $\{a_n\} = \left\{\frac{n}{2n+1}\right\}$

Does it ~~converge~~?

i.e., does $\boxed{f(x) = \frac{x}{2x+1}}$ have a HORIZONTAL ASYMPTOTE.

Start here, Tuesday

$a_n = \frac{n}{2n+1}$

$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$

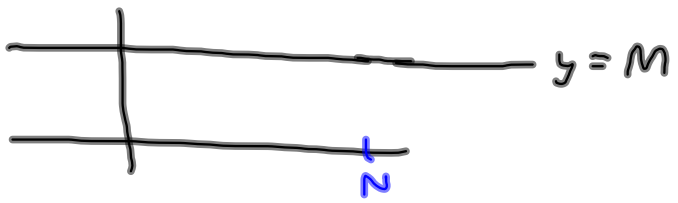
$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ $\qquad \frac{n}{2n+1} = \frac{\cancel{n}}{\cancel{n}(2+\frac{1}{n})} = \frac{1}{2+\frac{1}{n}} \xrightarrow{n\to\infty} \frac{1}{2}$

$\boxed{T3}$ $f(x) \xrightarrow{x\to\infty} L$ & $a_n = f(n)$

Then $a_n \xrightarrow{n\to\infty} L$ $\qquad a_n = \frac{2n-1}{3n+7}$ Look @ $f(x) = \frac{2x-1}{3x+7}$

$a_n \xrightarrow{n\to\infty} \frac{2}{3}$

$\boxed{D5}$ $a_n \xrightarrow{n \to \infty} \infty$ means if you give me $M > 0$, I can find $\underline{N \in \mathbb{N}}$ such that $a_n > M$ for every $n > N$.



$a_n = \ln(n)$ Claim: $a_n \xrightarrow{n \to \infty} \infty$

Proof: Let M be given. Let N be the least integer greater than e^M. Then, since $\ln(n)$ is an increasing function* of n, we have $\ln(n) > \ln(N) > M$, for every $n > N$.

* $\ln(x)$ is an increasing function of x: $\frac{dy}{dx} = \frac{1}{x} > 0$ for $x > 0$.

Scratch:
want
$\ln(n) > M$
$n > \underline{e^M \equiv N}$
(Details: want N to be the least integer bigger than e^M, if we insist that $N \in \mathbb{N}$)

Limit Laws $c \in \mathbb{R}$

$\lim(a_n + b_n)$, $\lim(a_n - b_n)$, $\lim(c\, a_n) = c \lim(a_n)$,
$\lim(a_n b_n)$, $\lim\left(\frac{a_n}{b_n}\right)^*$, $\lim(a_n^p)$

$\lim(a_n^p) = \left(\lim(a_n)\right)^p$

This all assumes $\lim(a_n), \lim(b_n) \ \exists$
and $* \lim(b_n) \neq 0$.

$\boxed{J7}$ If $a_n \xrightarrow{n\to\infty} L$ & f is cnt^s then

$\lim\limits_{n\to\infty} f(a_n) = f(L)$ *

$= f\left(\lim\limits_{n\to\infty} a_n\right)$

f is cnt^s means
$\lim\limits_{x\to c} f(x) = f(c)$

1, 2, 3, 5, 7 - 45 ODDS, 47, 49, 51, 53, 56? 57,
61, 63, 67!

Honest Effort.

5 mod 2 = 1

6 mod 2 = 0

A STAB at #44 - Above and beyond the instructions, we're trying to formulate a definition for a_n

$n=1 \quad \frac{1}{1},$

$n=3 \quad \frac{1}{2}$

$n=5 \quad \frac{1}{3}$

$n=2 \quad \frac{1}{3}$

$n=4 \quad \frac{1}{4}$

$n=6 \quad \frac{1}{5}$

$n=8 \quad \frac{1}{6}$

$n=10 \quad \frac{1}{7}$

$a_n = \begin{cases} \frac{1}{k} \text{ if } n = 2k-1 \\ \frac{1}{2k} \end{cases}$ Nope.

10 =

Determine whether it converges. If it converges, find its limit.

$1, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \frac{1}{5}, \frac{1}{7}, \frac{1}{6}, \frac{1}{8}, \frac{1}{7}, \frac{1}{9},$

Closed-form expression escapes me.
Good Bonus

Clearly seems to converge to zero

$a_{2\cdot 1} = a_2 = \frac{1}{3}$

$a_{2\cdot 2} = a_4 = \frac{1}{4}$

$a_{2\cdot 3} = a_6 = \frac{1}{5}$

$a_{2\cdot 4} = a_8 = \frac{1}{6}$

$a_{2\cdot 1-1} = a_1 = \frac{1}{1}$

$a_{2\cdot 2-1} = a_3 = \frac{1}{2}$

$a_{2\cdot 3-1} = a_5 = \frac{1}{3}$

$a_{2\cdot 4-1} \quad a_7 = \frac{1}{4}$

$a_n = \begin{cases} \frac{1}{k+2} & n = 2k \\ \frac{1}{k} & n = 2k-1 \end{cases}$

$\boxed{y'' - 5y' + 6y} = 0$ $\boxed{y'(0) = y(0) = 1}$

Characteristic Polynomial

$D^2 = \frac{d^2}{dx^2}$

$(D^2 - 5D + 6)y = 0$

$D^2 - 5D + 6 = 0$

$(D-3)(D-2) = 0$

$D = 2, 3$

$y' = 2c_1e^{2t} + 3c_2e^{3t}$

$y'' = 4c_1e^{2t} + 9c_2e^{3t}$

$\boxed{y = c_1e^{2t} + c_2e^{3t}}$

Ordinary Algebra

General Solution

Particular Solution

$y'' - 5y' + 6y =$

$4c_1e^{2t} + 9c_2e^{3t}$

$-10c_1e^{2t} - 15c_2e^{3t}$

$+6c_1e^{2t} + 6c_2e^{3t} = 0$

$y(0) = 1 = c_1 + c_2 = 1 \Longrightarrow c_1 = 1 - c_2$

$y'(0) = 1 = 2c_1 + 3c_2 = 1 \Longrightarrow$

$2(1 - c_2) + 3c_2 = 1$

$2 - 2c_2 + 3c_2 = 1$

$2 + c_2 = 1$

$\boxed{c_2 = -1}$

$\boxed{c_1 = 1 - (-1) = 2}$

$\boxed{y = 2e^{2t} - e^{3t}}$

Satisfies the eq'n & initial conditions.

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$y' = x + 2y$ Solve it

$\frac{dy}{dx} = x + 2y$

$y' - 2y = x$

$P(x) = -2$

$e^{\int p(x)\,dx} = e^{-2x}$

$\int -2\,dx = -2x + C$

$\underbrace{e^{-2x}y' - 2e^{-2x}y}_{(e^{-2x}y)'} = e^{-2x}\cdot x$

$(e^{-2x}y)' = xe^{-2x}$

$\frac{d\,(e^{-2x}y)}{dx} = xe^{-2x}$

$\int d\,(e^{-2x}y) = \int xe^{-2x}dx$

$u = x,\ du = dx$

$dv = e^{-2x}dx$

$v = -\frac{1}{2}e^{-2x}$

$e^{-2x}y = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}dx$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

$\boxed{y = -\frac{1}{2}x - \frac{1}{4} + Ce^{2x}}$

$\int \sec^3\theta \, d\theta$

$\int \sqrt{1+u^2}\, du$

$y = -\frac{1}{2}x - \frac{1}{4} + ce^{2x}$

$$y' = x + 2y$$

Not recommended

Let $u = x + 2y$

Then $u' = 1 + 2y'$

$u' - 1 = 2y'$

$\frac{u'-1}{2} = y'$

$$\frac{u'-1}{2} = u$$

$$u' - 1 = 2u$$

$$u' = 2u + 1$$

$$\frac{du}{dx} = 2u + 1$$

$$\frac{1}{2}\int \frac{2du}{2u+1} = \int dx$$

$v = 2u + 1$

$dv = 2du$

$\int \frac{dv}{v}$

$$\frac{1}{2}\ln|2u+1| = x + C$$

$$\ln(2u+1) = 2x + 2C$$

$$2u+1 = e^{2x+2C} = e^{2C}e^{2x} = Ke^{2x}$$

$$2u = Ke^{2x} - 1$$

$$u = \frac{K}{2}e^{2x} - \frac{1}{2}$$

$$x + 2y = \frac{K}{2}e^{2x} - \frac{1}{2}$$

$$2y = \frac{K}{2}e^{2x} - \frac{1}{2} - x$$

$$y = \frac{K}{4}e^{2x} - \frac{1}{4} - \frac{1}{2}x$$

C

Forgot constant of integration

$$y = -\frac{1}{2}x - \frac{1}{4} + ce^{2x}$$

$$\int y \sqrt[3]{1+25y^4}\, dy$$

$u = 5y^2$
$du = 10y\,dy$

$$= \frac{1}{10}\int 10y \sqrt[3]{1+(5y^2)^2}\, dy$$

$$= \frac{1}{10}\int \sqrt[3]{1+u^2}\, du$$

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