11.4 #s 1, 4, 7, 10, 13, 15, 20, 26, 31, 35, 39, 42, 45, 52

11.4 AREAS AND LENGTHS IN POLAR COORDINATES

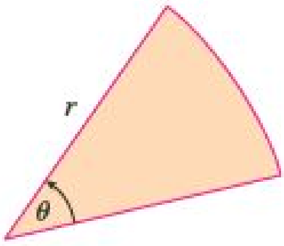


FIGURE 1

$$A = \frac{1}{2}r^2\theta$$

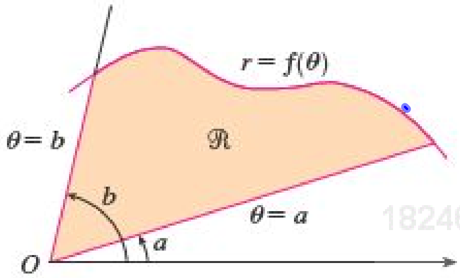


FIGURE 2

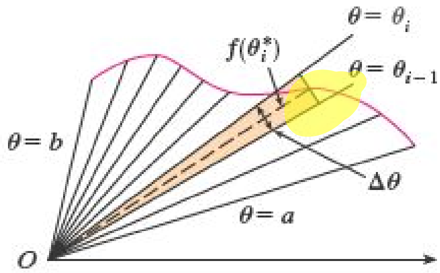


FIGURE 3

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$$

$$\boxed{A = \int_a^b \frac{1}{2}r^2\,d\theta}$$

$$\boxed{A = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta}$$

The general idea:

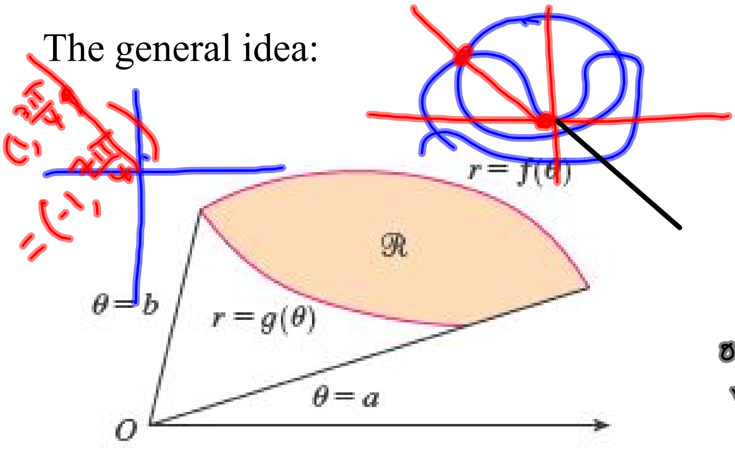


FIGURE 6

Area between two functions

$$\int_a^b (f(x) - g(x))\,dx$$

Sometimes the limits of integration don't match up. θ might be different for g at the intersection

Area of $\mathcal{R} = A = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2\,d\theta = \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right)d\theta$

may be inappropriate.

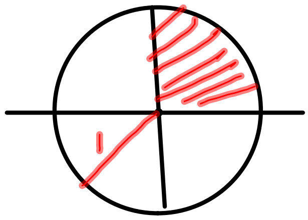
CAUTION

the origin has no single representation in polar coordinates that satisfies both equations."

Algebraic techniques won't always reveal a key point. One trick: Substitute - *r* for *r* since sometimes the point of intersection will have an angle that's, for instance, π radians (or some multiple thereof) away from the other angle, and *r* is negative!

But the *best* way is to be clear on what the picture is and determine the relevant values for each function. Technology is a HUGE help in this!!! (Wish I'd had it when *I* took calculus. This stuff would've been a *whole* lot less mysterious!)

Motivation:
what's the area enclosed by the circle $r=1$?



Polar coords:

$$4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 1^2\,d\theta = 2\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\cdot\frac{\pi}{2} = \pi = \pi r^2 \text{ as expected: } \pi\cdot 1^2$$

Rectangular:

$$x^2+y^2=r^2$$
$$y^2=r^2-x^2$$
$$y=\pm\sqrt{r^2-x^2} \quad \text{take piece in QI}$$
$$y=+\sqrt{r^2-x^2} = \sqrt{1-x^2}$$

$$\text{Area} = 4\int_0^1 \sqrt{1-x^2}\,dx$$

$x = a\sin\theta = \sin\theta$
$dx = \cos\theta\,d\theta$
$x=0=\sin\theta \to \theta=0$
$x=1=\sin\theta \to \theta=\frac{\pi}{2}$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(1-\sin^2\theta)}\cos\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \cos\theta\;\cos\theta\,d\theta$$

$|\cos\theta| = \cos\theta$ on $[0,\frac{\pi}{2}]$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$u=2\theta$

$$= 2\int_0^{\frac{\pi}{2}} d\theta + \frac{2}{2}\int_0^{\frac{\pi}{2}} 2\cos(2\theta)\,d\theta$$

works, but a pain.

$$= 2\left[\theta\right]_0^{\frac{\pi}{2}} + \sin(2\theta)\Big]_0^{\frac{\pi}{2}} = \pi$$

$$\sin(2\cdot\tfrac{\pi}{2}) - \sin(2\cdot 0) = \sin(\pi)-\sin(0) = 0$$

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

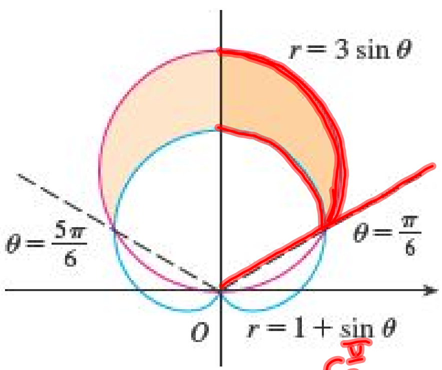
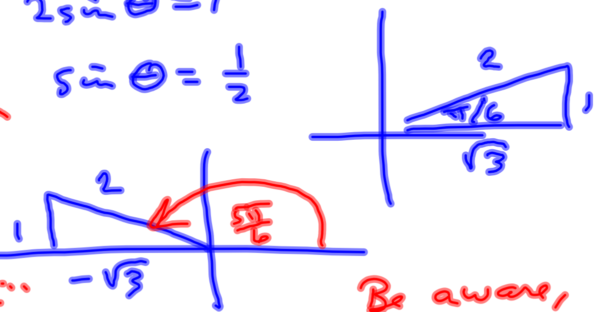


FIGURE 5

$$3\sin\theta = 1 + \sin\theta$$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$



$2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(3\sin^2\theta) - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cdots$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1+\sin\theta)^2\,d\theta$$

Be aware, the intersections might be obtained by different values of θ.

$$= \frac{1}{2}\cdot 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right]d\theta$$

$$= 8\int_{\frac{\pi}{6}}^{\pi/2}\sin^2\theta\,d\theta - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1 + 2\sin\theta)\,d\theta, \text{ etc.}$$

$r = 3|\sin\theta|$

is a figure 8!

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

5–8 Find the area of the shaded region.

9–14 Sketch the curve and find the area that it encloses.

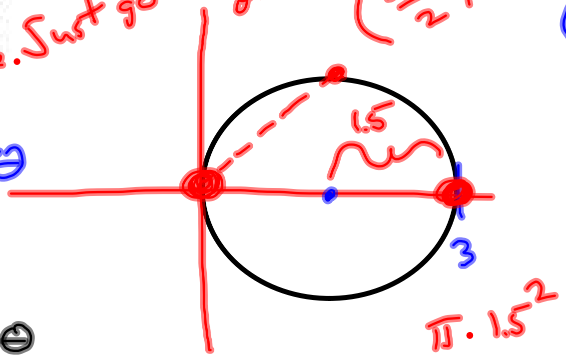
9. $r = 3\cos\theta$

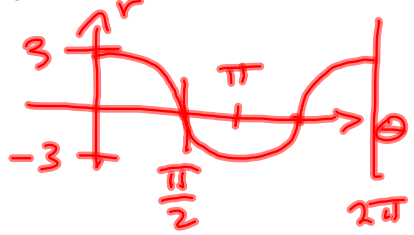
Nope. Just go 'round once.

$\left(\frac{3\sqrt{2}}{2}, \frac{\pi}{4}\right)$

$(1.5)^2\pi = \text{Area}$

$\frac{1}{2}\int_0^{2\pi}(3\cos\theta)^2\,d\theta$





$= \frac{1}{2}\int_0^{\pi} 9\cos^2\theta\,d\theta$

$\frac{9}{2}\int_0^{\pi}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{9}{4}\int_0^{\pi} d\theta + \frac{9}{8}\int_0^{\pi}\cos(2\theta)\,2\,d\theta$

$= \frac{9}{4}\theta\Big]_0^{\pi} + \frac{9}{8}\Big[\sin(2\theta)\Big]_0^{\pi}$

$= \frac{9}{4}\pi = \left(\frac{3}{2}\right)^2\pi = \boxed{(1.5)^2\pi}$

17–21 Find the area of the region enclosed by one loop of the curve.

18. $r = 4\sin 3\theta$

pole — r by $-r$

polar axis — θ by $-\theta$

$\theta = \frac{\pi}{2}$ — θ by $\pi - \theta$

I still always end up graphing an entire 2*pi's worth of the graph, unless I can clearly see that I'm retracing my steps. The symmetry considerations, for me, are more confirmation of my graph than a real assistance to graphing, itself. I just have to plug on through at least once around, just to make sure I didn't miss any cutesie little loops.

Symmetry:

pole. $-r = 4\sin(3\theta)$

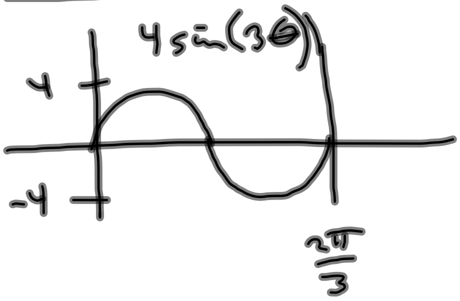
$r = -4\sin(3\theta)$ not the same. $r = 4\sin(3\theta)$

polar axis: $r = 4\sin(3(-\theta)) = 4\sin(-3\theta) = -4\sin(3\theta)$ Nerp.

$\theta = \frac{\pi}{2}$: $r = 4\sin(3(\pi - \theta)) = 4\sin(3\pi - 3\theta)$

$= 4\sin(3\pi)\cos(3\theta) - 4\sin(3\theta)\cos(3\pi)$

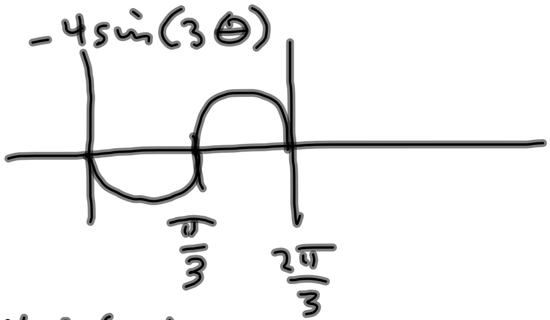
$= 0 + 4\sin(3\theta)$ Same.



$4\sin(3(\pi - \theta))$

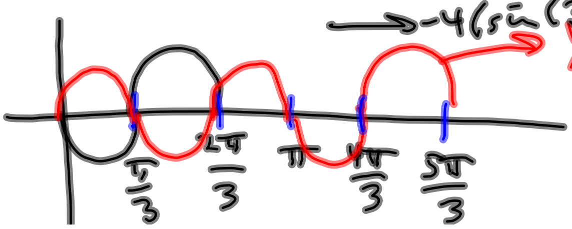
$= 4\sin(-3(\theta - \pi))$

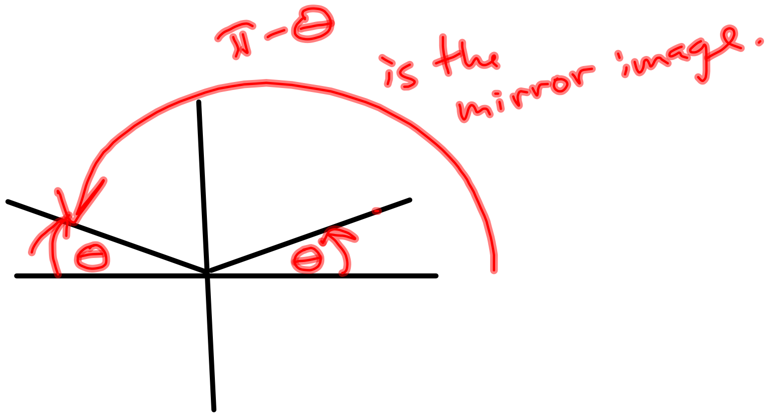
$= -4\sin(3(\theta - \pi))$



$\sin\theta \to \sin(3\theta) \to -4\sin(3\theta)$

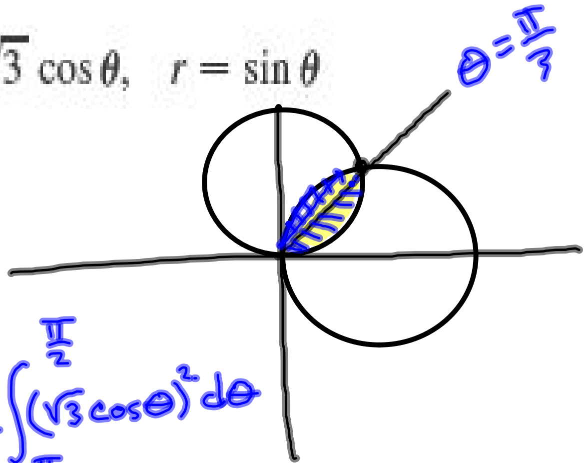
$\to -4(\sin(3(\theta - \pi)))$ Yes. Same as original picture




π - θ
is the mirror image.
θ
θ

29–34 Find the area of the region that lies inside both curves.

29. $r = \sqrt{3}\cos\theta$, $r = \sin\theta$



$\sin\theta = \sqrt{3}\cos\theta$

$\tan\theta = \sqrt{3}$

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\theta)^2\, d\theta$$

$$+ \frac{1}{2}\int_{0}^{\frac{\pi}{3}} (\sin\theta)^2\, d\theta$$

32. $r = 3 + 2\cos\theta, \quad r = 3 + 2\sin\theta$

36. Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

37–42 Find all points of intersection of the given curves.

$\boxed{41.}$ $r = \sin\theta, \quad r = \sin 2\theta$

ARC LENGTH

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

By Theorem 11.2.6
Arc Length is given by L :

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

45–48 Find the exact length of the polar curve.

46. $r = e^{2\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$

Oct 27-11:17 AM

49–52 Use a calculator to find the length of the curve correct to four decimal places.

50. $r = 4 \sin 3\theta$