8.3 TRIGONOMETRIC SUBSTITUTION

Recall: Substitution Rule

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Today, we do this:

$$\int f(u)\,du = \int f(g(x))\,g'(x)\,dx$$

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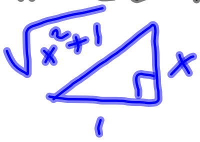
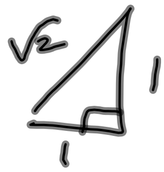
$$\int_0^1 \sqrt{x^2+1}\,dx \qquad a=1 \qquad x = a\tan\theta = \tan\theta = x$$

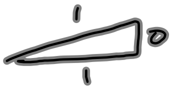


$$x = \tan\theta = 1$$

$$\theta = \arctan(1) = \frac{\pi}{4}$$



$$x = \tan\theta = 0 \Longrightarrow \theta = 0$$



$$dx = \sec^2\theta\,d\theta$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan^2\theta+1}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{\tan^2\theta+1}\,\sec^2\theta\,d\theta$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} |\sec\theta|\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta =$$

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See Example 8.2.8

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \int_0^{\frac{\pi}{4}} \sec\theta \sec^2\theta \, d\theta$$

$u = \sec\theta \Rightarrow du = \sec\theta\tan\theta \, d\theta$

$dv = \sec^2\theta \, d\theta$

$v = \tan\theta$

$$= uv - \int v\,du = uv - I_1$$

$$= \sec\theta\tan\theta \Big|_0^{\frac{\pi}{4}} - I_1 = \sec\tfrac{\pi}{4}\tan\tfrac{\pi}{4} - I_1 = \sqrt{2}\cdot 1 - I_1 = \sqrt{2} - I_1$$

$$I_1 = \int_0^{\frac{\pi}{4}} \tan\theta \sec\theta \tan\theta \, d\theta = \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta - \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta = \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta - \ln|\sec\theta + \tan\theta| \Big]_0^{\frac{\pi}{4}}$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta - \ln|\sqrt{2} + 1| = I_1$$

Put it together:

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \sqrt{2} - I_1 = \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta + \ln|\sqrt{2} + 1|$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \frac{\sqrt{2} + \ln|\sqrt{2} + 1|}{2}$$

24. $\displaystyle\int \frac{dt}{\sqrt{t^2 - 6t + 13}}$

$t^2 - 6t + 13$

$= t^2 - 6t + 3^2 - 9 + 13$

$= (t-3)^2 + 4$

$u = t - 3 \Longrightarrow du = dt$

$$= \int \frac{dt}{\sqrt{(t-3)^2 + 4}}$$

$$= \int \frac{du}{\sqrt{u^2+4}} = \int \frac{du}{\sqrt{u^2+2^2}}$$

$a = 2$

Let $u = a\tan\theta$

$= 2\tan\theta$

$du = 2\sec^2\theta\, d\theta$

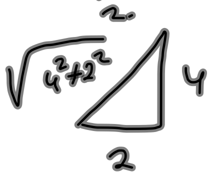
$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{(2\tan\theta)^2 + 2^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{2^2\tan^2\theta + 2^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{2^2(\tan^2\theta + 1)}}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{2\sqrt{\tan^2\theta + 1}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta}*$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$u = 2\tan\theta$

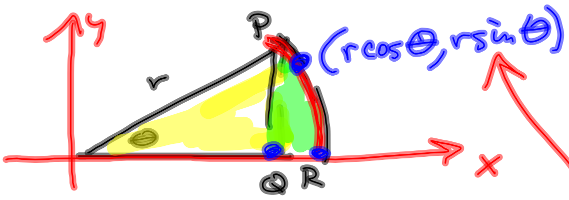
$\frac{u}{2} = \tan\theta$



$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C$$

$$= \ln\left|\frac{\sqrt{t^2-6t+13}}{2} + \frac{t-3}{2}\right| + C$$

$A = \frac{1}{2} r^2 \theta$



Area = Triangle +

$x^2+y^2=r^2$

$\Rightarrow y^2 = r^2 - x^2$

$\Rightarrow y = \pm\sqrt{r^2-x^2}$

$\Rightarrow y = \sqrt{r^2-x^2}$ is what we want.

$= \sqrt{r^2-(r\cos\theta)^2} = \sqrt{r^2(1-\cos^2\theta)} = \sqrt{r^2\sin^2\theta} = \boxed{r\sin\theta}$

$$\text{Triangle} + \int_Q^R \sqrt{r^2-x^2}\,dx = \frac{1}{2} r\sin\theta\, r\cos\theta + \int_{r\cos\theta}^{r} \sqrt{r^2-x^2}\,dx$$

$$= \frac{r^2}{2}\sin\theta\cos\theta + I_1$$

$$I_1 = \int_{r\cos\theta}^{r} \sqrt{r^2-x^2}\,dx$$

Let $x = r\sin\theta$
$dx = r\cos\theta\, d\theta$

$x = r\sin\theta = r\cos\theta \Rightarrow$
$1 = \cot\theta$
$\theta = \frac{\pi}{4}$

$x = r\sin\theta = r$
$\sin\theta = 1$
$\theta = \frac{\pi}{2}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cos\theta\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\left(\frac{1}{2}(1+\cos(2\theta))\right)d\theta$$

$$= \frac{r^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos(2\theta))\,d\theta = \frac{r^2}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) + \frac{r^2}{2}\cdot\frac{1}{2}\Big[\sin(2\theta)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{r^2}{2}\left(\frac{\pi}{4}\right) + \frac{r^2}{4}\Big[0-1\Big] = \frac{\pi r^2}{8} - \frac{r^2}{4}$$

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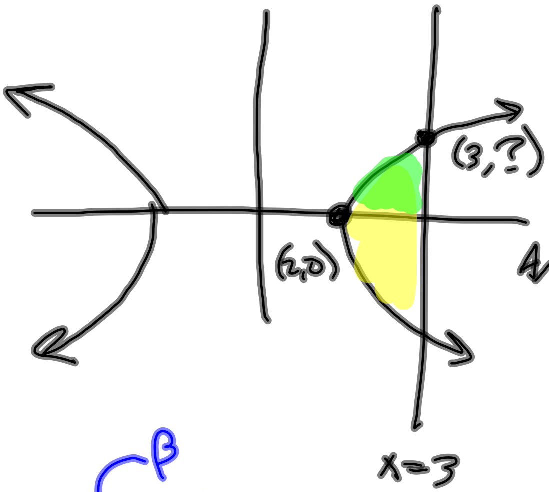
34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

$\frac{x^2}{4} - \frac{y^2}{9} = 1$

$\frac{y^2}{9} = \frac{x^2}{4} - 1 = \frac{x^2-4}{4}$

$y^2 = \frac{9}{4}(x^2-4)$

$y = \pm\frac{3}{2}\sqrt{x^2-4}$



$\text{Area} = 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx$

$\cancel{x} = 2\sec\theta$

$\frac{x}{2} = \sec\theta$

$x = 2$

$2 = 2\sec\theta$

$\sec\theta = 1$

$\text{arcsec}\left(\frac{3}{2}\right)$

$3 = 2\sec\theta$

$\sec\theta = \frac{3}{2}$

$\theta = \text{arcsec}\left(\frac{3}{2}\right) = \beta$

$\int_0^\beta \text{etc.}$