8.3 TRIGONOMETRIC SUBSTITUTION

Recall: Substitution Rule

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

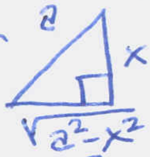
Today, we do this:

$$\int f(u)\,du = \int f(g(x))\,g'(x)\,dx$$

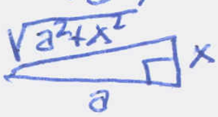
Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Trig Substitution

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$



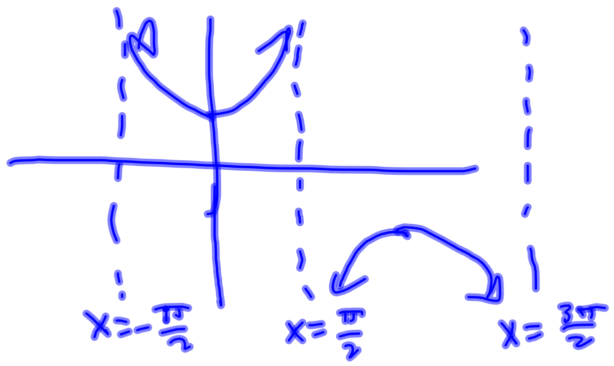
$\sqrt{a^2+x^2}$ $\quad x = a\tan\theta$



$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$

$$\sqrt{\sec^2 x} = |\sec x|$$

It will be your job to decide/know/guess if it's sec x or − sec x coming out of the absolute value, based on the domain in question.



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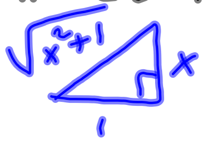
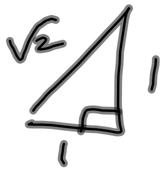
$\int_0^1 \sqrt{x^2+1}\,dx$ $\quad a=1 \quad x = a\tan\theta = \tan\theta = x$

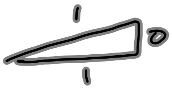


$x = \tan\theta = 1$

$\theta = \arctan(1) = \frac{\pi}{4}$



$x = \tan\theta = 0 \Rightarrow \theta = 0$



$dx = \sec^2\theta\,d\theta$

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan^2\theta+1}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{\tan^2\theta+1}\,\sec^2\theta\,d\theta$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} |\sec\theta|\,\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \int_0^{\frac{\pi}{4}} (\tan^2\theta+1)\sec\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec\theta\tan^2\theta + \sec\theta)\,d\theta$$

$$\int \sec^2\theta\,\sec\theta\,d\theta$$

$= \int \sec\theta\,(\sec^2\theta - 1)\,d\theta$ leads nowhere.

Surely not an integration by parts?

$$\int \frac{\sin^2\theta}{\cos^3\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos^3\theta}\,d\theta = \int \sec^3\theta - \sec\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec\theta \tan^2\theta + \sec\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec^3\theta - \sec\theta \ + \sec\theta)\, d\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$$

$u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$dv = \sec^2\theta \, d\theta$

$v = \tan\theta$

$$= \left[\sec\theta \tan\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta \tan^2\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta \quad = \int \sec\theta \sec^2\theta \, d\theta$$

$u = \sec\theta$ $\quad dv = \sec^2\theta \, d\theta$

$du = \sec\theta\tan\theta \, d\theta$ $\quad v = \tan\theta$

$$= \sec\theta\tan\theta - \int \tan^2\theta \sec\theta \, d\theta$$

$$\int e^x \sin x \, dx \ ?$$

24. $\displaystyle\int \frac{dt}{\sqrt{t^2 - 6t + 13}}$

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.