

8.3 TRIGONOMETRIC SUBSTITUTION

Recall: Substitution Rule

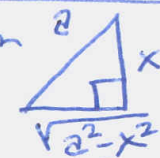
$$\int f(g(x))g'(x)dx = \int f(u)du$$

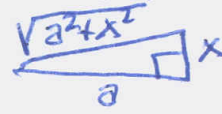
Today, we do this:


$$\int f(u)du = \int f(g(x))g'(x)dx$$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Trig Substitution

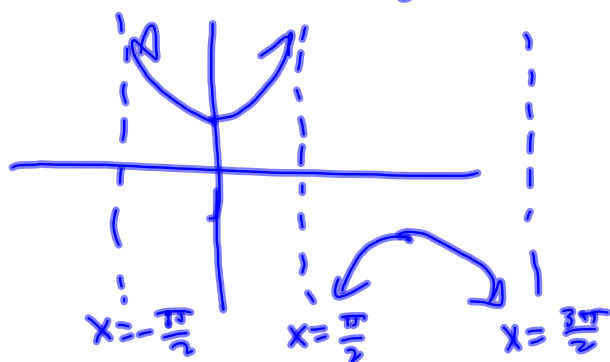
$\sqrt{a^2 - x^2}$ $x = a \sin \theta$ 

$\sqrt{a^2 + x^2}$ $x = a \tan \theta$ 

$\sqrt{x^2 - a^2}$ $x = a \sec \theta$ 

$$\sqrt{\sec^2 x} = |\sec x|$$

It will be your job to decide/know/guess if it's $\sec x$ or $-\sec x$ coming out of the absolute value, based on the domain in question.



$$\int_0^1 \sqrt{x^2+1} dx \quad a=1 \quad x = a \tan \theta = \tan \theta = x$$



$$x = \tan \theta = 1$$

$$\theta = \arctan(1) = \frac{\pi}{4}$$

$$x = \tan \theta = 0 \Rightarrow \theta = 0$$



$$dx = \sec^2 \theta d\theta$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan^2 \theta + 1} dx = \int_0^{\frac{\pi}{4}} \sqrt{\tan^2 \theta + 1} \sec^2 \theta d\theta$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\sec^2 \theta} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{4}} |\sec \theta| \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta = \int_0^{\frac{\pi}{4}} (\tan^2 \theta + 1) \sec \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec \theta \tan^2 \theta + \sec \theta) d\theta$$

$$\int \sec^2 \theta \sec \theta d\theta$$

$$= \int \sec \theta (\sec^2 \theta - 1) d\theta \quad \text{leads nowhere.}$$

• Surely not an integration by parts?

$$\int \frac{\sin^2 \theta}{\cos^3 \theta} d\theta = \int \frac{1 - \cos^2 \theta}{\cos^3 \theta} d\theta = \int \sec^3 \theta - \sec \theta$$

$$\int \sec^3 \theta d\theta = \int \sec \theta \sec^2 \theta d\theta$$

$u = \sec \theta$ $dv = \sec^2 \theta d\theta$
 $du = \sec \theta \tan \theta d\theta$ $v = \tan \theta$

$$= \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta d\theta$$

$$\int e^x \sin x dx ?$$

$$24. \int \frac{dt}{\sqrt{t^2 - 6t + 13}}$$

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.