#13 (a) $\cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2}$ Last Thursday.

(*) $f'(x) = -\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0$

in service of $\frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$ in (b)

All we need from (*) is f is constant.
And since it is,

$f(1) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$, so $f(x) = \frac{\pi}{2}$

So that's good, but you USED $\frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$ which you've not yet proven! So hand-slap.

Still I liked the method's cleverness.

Recall

Prove $A = A_0 e^{rt}$ is the formula for continuous compounding.

Compounded m times per annum:

$$A(t) = A_0\left(1+\frac{r}{m}\right)^{mt}$$

$$= A_0\left(1+\frac{r}{m}\right)^{\frac{m}{r}\cdot rt} = A_0\left(\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right)^{rt}$$

If we can show that $\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{\frac{m}{r}} = e$, we're done.

Let $y = \left(1+\frac{r}{m}\right)^{\frac{m}{r}}$. Then

$$\ln(y) = \frac{m}{r}\ln\left(1+\frac{r}{m}\right)$$

$$\lim_{m\to\infty}\ln(y) = \lim_{m\to\infty}\left(\frac{m}{r}\ln\left(1+\frac{r}{m}\right)\right)$$

$\infty \cdot 0$

$$= \lim_{m\to\infty}\frac{\ln\left(1+\frac{r}{m}\right)}{\frac{r}{m}} \overset{L'H}{=} \lim_{m\to\infty}\frac{\frac{-\frac{r}{m^2}}{1+\frac{r}{m}}}{-\frac{r}{m^2}}$$

$\frac{0}{0}$

$$= \lim_{m\to\infty}\frac{\frac{1}{1+\frac{r}{m}}}{1} = \lim_{m\to\infty}\left(\frac{1}{1+\underbrace{\frac{r}{m}}_{\to 0}}\right) = 1$$

$$\Rightarrow \ln(y) \xrightarrow{m\to\infty} 1 \quad \Rightarrow$$

$$y \xrightarrow{m\to\infty} e^1 = e \quad \blacksquare$$

Find $f'(x)$ if $f(x) = \int_1^{\sqrt{x}} \cos^2(\theta)\sqrt{22\theta}\, d\theta$

FTC II with a twist of CHAIN RULE

$$\cos^2(\sqrt{x})\sqrt{22\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\int_2^x f(t)\,dt = f(x)$$ This is a function of x

$$f'(x) = 3x+2 = \frac{df}{dx}$$

$$\frac{d}{dx}\left[f(\sqrt{x})\right] = 3\sqrt{x}$$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx$$

$u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$u = 1 - x^4$

$du = -4x^3\,dx$

where's the x^3? They ain't one

$$= \frac{1}{2}\int \frac{2\,x\,dx}{\sqrt{1-(x^2)^2}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin(u) + C$$

$$= \boxed{\frac{1}{2}\arcsin(x^2) + C}$$

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$$\int \ln(\cos(x))\tan(x)\,dx \qquad \frac{d}{dx}\left[\ln(f(x))\right] = \frac{f'(x)}{f(x)}$$

Let $u = \ln(\cos(x))$

$$du = \frac{-\sin(x)}{\cos(x)}\,dx = -\tan(x)\,dx$$

$$= -\int \ln(\cos(x))(-\tan(x))\,dx$$

$$= -\int u\,du = -\frac{u^2}{2} + C = -\frac{(\ln(\cos(x)))^2}{2} + C$$

$f(x) = x + x^2 + e^x$. Find $(f^{-1})'(1)$

Want $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$, so we need $f^{-1}(1)$

Need $f(x) = x + x^2 + e^x = 1$ Solve for x.

$0 + 0^2 + e^0 = 1$ Sweet!

By Inspection.

$f^{-1}(1) = 0$

we're this far: $\dfrac{1}{f'(0)}$

Need $f'(x) = 1 + 2x + e^x$ to get $f'(0)$

$\Rightarrow f'(0) = 1 + 0 + 1$

$= 2$

This gives $(f^{-1})'(1) = \dfrac{1}{2}$

$\tanh(x) = \frac{3}{5}$ Find the value of the other 5...

$\coth(x) = \frac{5}{3}$ for free

No Calculator.

$\boxed{1 - \tanh^2(x) = \operatorname{sech}^2(x)}$

$1 - \frac{9}{25} = \frac{16}{25} = \operatorname{sech}^2(x) \Longrightarrow \operatorname{sech}(x) = \frac{4}{5} \Longrightarrow$

$\cosh(x) = \frac{5}{4}$

$\cosh^2(x) - \sinh^2(x) = 1$

Requires Pythagorean Identities

$$\coth(x) = \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x)$$

Matthew asks about sign issues with $\sinh(x)$, here.

Steve brilliantly replies:
It follows the sign of the $\tanh(x)$