L'Hôpital

7.8 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

$\frac{0}{0}$

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

if $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$,

indeterminate form of type $\frac{0}{0}$.

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \frac{1}{2}$$

1 $$\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$$

$\frac{0}{0}$ But how?

2 $$\lim_{x \to \infty} \frac{\ln x}{x - 1} = 0$$

$\frac{\infty}{\infty}$

Other indeterminate forms:

$0 \cdot \infty$

$\infty - \infty$

0^0

∞^0

1^∞

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L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

crucial in proof of $\frac{d}{dx}[\sin(x)] = \cos(x)$

See back of book.

Figure 1 is an attempt to help your intuition, somewhat. It kind of makes sense that if numerator and denominator both approach zero, the actual limit of the quotient probably is controlled by how *fast* each is approaching zero. In other words, their slopes at that limiting value can easily be imagined to have some sort of effect on the limit of the quotient.

$$\lim_{x \to 0} \frac{\sin(x)}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

$\frac{0}{0}$ ucky.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \frac{\sin(x)(\cos(h) - 1) + \sin(h)\cos(x)}{h}$$

$$= \frac{\cos(h) - 1}{h}\sin(x) + \frac{\sin(h)}{h}\cos(x) \xrightarrow{h \to 0} \cos(x)$$

$$\frac{\cos(h)-1}{h} \xrightarrow{h \to 0} 0 \qquad \frac{\sin(h)}{h} \xrightarrow{h \to 0} 1$$

These fussy bits were hard, before, but are a cinch with L'Hôpital!

$$\lim_{x\to 1} \frac{\ln(x)}{\sin(\pi x)} \overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{\pi\cos(\pi x)} = \frac{\frac{1}{1}}{\pi(-1)} = \frac{1}{-\pi} = -\frac{1}{\pi}$$

$\frac{0}{0}$ $\qquad \frac{\frac{1}{1}}{}$

(5-3) $\lim_{x\to 0^+} x^{x^2}$ $\qquad \lim_{x\to\infty} \frac{x^2}{x} = \infty$

0^0

$y = x^{x^2}$

$\ln(y) = \ln(x^{x^2}) = x^2 \cdot \ln(x)$

$0 \cdot (-\infty)$

$\frac{\ln(x)}{\frac{1}{x^2}} \overset{L'H}{=}$ Sloppy notation.

$\frac{-\infty}{+\infty}$

we evaluate

$$\lim_{x\to 0^+}(\ln(y)) = \lim_{x\to 0^+}\left(x^2 \cdot \ln(x)\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\ln(x)}{\frac{1}{x^2}}\right) = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x\to 0^+}\left(\frac{1}{x}\cdot\left(-\frac{x^3}{2}\right)\right)$$

$$= \lim_{x\to 0^+}\left(-\frac{x^2}{2}\right) = 0.$$

$\therefore \lim_{x\to 0^+} \ln(y) = 0 \Longrightarrow \lim_{x\to 0^+} y = e^0 = 1$

i.e., $\boxed{\lim_{x\to 0^+} x^{x^2} = 1}$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$$ similar to previous

1^{∞}

Indeterminate

Recall: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$

$$y = \left(1+\frac{1}{n}\right)^{n}$$

$$\ln(y) = n\ln\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}(\ln(y)) = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$\infty\cdot 0$ $\qquad \frac{0}{0}$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{-\frac{1}{n^2}}{1+\frac{1}{n}}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

$\therefore \lim_{n\to\infty}\ln(y) = 1$

$\therefore \lim_{n\to\infty} y = e^{1} = e$

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin(5x)} = \lim_{x\to 0} \frac{3\sec^2(3x)}{5\cos(5x)} = \frac{3\cdot 1}{5\cdot 1} = \frac{3}{5}$$

$$\frac{0}{0}$$

3 CAUCHY'S MEAN VALUE THEOREM Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Generalized MVT

NOTE: If $g(x) = x$, then $\frac{f'(c)}{g'(c)} = \frac{f'(c)}{1} = f'(c) = \frac{f(b) - f(a)}{b - a}$