7.7 HYPERBOLIC FUNCTIONS

7.7 #s 1 – 6, 8, 10, 18, 19*, 20, 24a, 26, 28c, 34, 40, 46, **52** (Class project - show all steps, everyone get involved.)
#s 1 – 6 (and 'most all the assignment), think in terms of $\frac{e^x - e^x}{2}$.

Also remember that an often-practical way to evaluate an inverse hyperbolic/trigonometric function is to solve a hyperbolic/trigonometric equation, and check your domains (off a cheatsheet). We will collaborate on a cheatsheet for the class to use.
#s 11, 12 are not assigned, but look like good cheatsheet material.
#19 is a nice bonus problem. Answer's in the back. Looks like it oughta have a nice, clean induction proof.

DEFINITION OF THE HYPERBOLIC FUNCTIONS

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

catenary

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

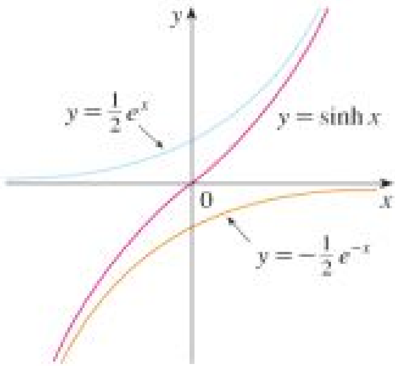

FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

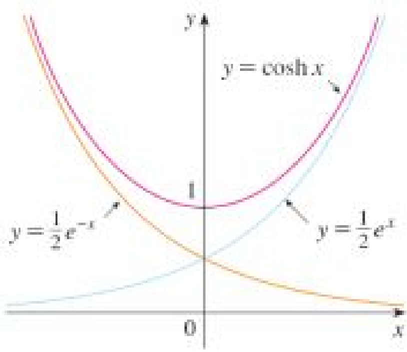

FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

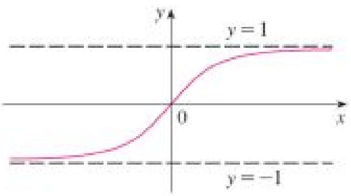

FIGURE 3
$y = \tanh x$

HYPERBOLIC IDENTITIES

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\cos 2x = 1 - 2\sin^2 x$$

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1 DERIVATIVES OF HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

2

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \text{ and } y \geqslant 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

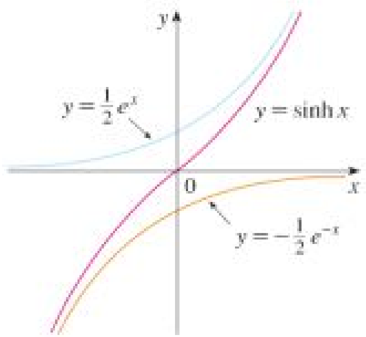

FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

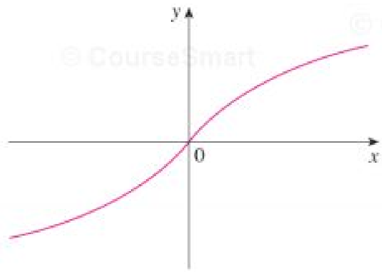

FIGURE 8 $y = \sinh^{-1}x$
domain = $\mathbb{R}$ range = $\mathbb{R}$

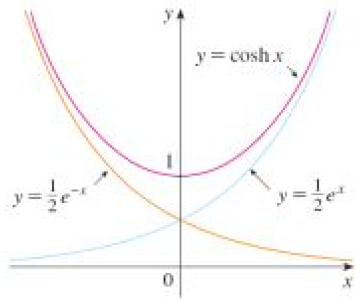

FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

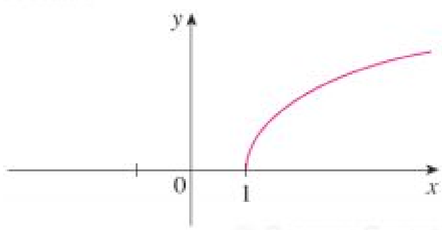

FIGURE 9 $y = \cosh^{-1}x$
domain = $[1, \infty)$ range = $[0, \infty)$

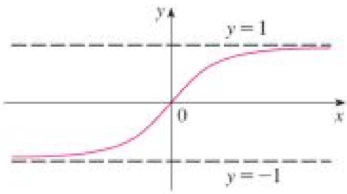

FIGURE 3
$y = \tanh x$

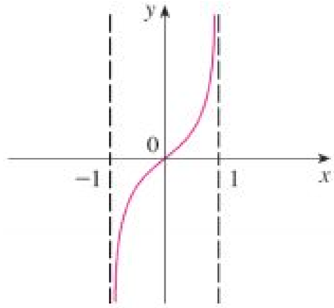

FIGURE 10 $y = \tanh^{-1}x$
domain = $(-1, 1)$ range = $\mathbb{R}$

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$$\boxed{3} \qquad \sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right) \qquad x \in \mathbb{R}$$

$$\boxed{4} \qquad \cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geqslant 1$$

$$\boxed{5} \qquad \tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$

The proof of the first of these identities is given in Example 3. This is worth going over, with a slightly different way of getting started.

$$y = \sinh^{-1}(x) \Longrightarrow \sinh(y) = x$$

$$x = \sinh(y) = \frac{e^y - e^{-y}}{2} \Longrightarrow 2x = e^y - e^{-y}$$

$$2xe^y = e^{2y} - 1 \qquad\qquad (e^y)^2 = e^{2y}$$

$$e^{2y} - 2xe^y - 1 = 0 \qquad\qquad e^{y^2} \neq e^{2y}$$

$$u^2 - 2xu - 1 = 0$$

$$b^2 - 4ac = (-2x)^2 - 4(1)(-1)$$

$$= 4x^2 + 4$$

$$u = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2} = \frac{2\left(x \pm \sqrt{x^2+1}\right)}{2}$$

$$= x \pm \sqrt{x^2+1} = u = e^y$$

$$y = \ln\left(x \pm \sqrt{x^2+1}\right), \text{ but } e^y > 0 > x - \sqrt{x^2+1}$$

$$\text{so } \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

This argument is harder to use for $\cosh^{-1}(x)$, because we don't have the "but"

But get me $\boxed{\text{this far}}$ on $\cosh^{-1}(x)$ proof & I'll be pretty happy.

6 **DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS**

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$$

See Example 4. It's the sort of argument one might be expected to make for the derivative of the inverse hyperbolic cosine.

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

To remember this or to help keep it straight, look at the definitions of sinh x and cosh x and notice that $-1 < \tanh x < 1$, whereas $|\coth x| > 1$.

Something you won't see in the textbook:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

To read more on this, Google "exponential form of sine" and follow the *Wolfram* link.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$\coth(x) = \frac{6}{5}$ Find the other on

$\tanh(x) = \frac{5}{6} = \frac{\sinh(x)}{\cosh(x)}$

$1 - \tanh^2(x) = \operatorname{sech}^2(x)$

$1 - \frac{25}{36} = \operatorname{sech}^2(x)$

$\frac{11}{36} = \operatorname{sech}^2(x)$

$\operatorname{sech}(x) = \pm\sqrt{\frac{11}{36}} = \pm\frac{\sqrt{11}}{6} = \frac{1}{\cosh(x)} = \frac{1}{\frac{e^x+e^{-x}}{2}} > 0$

so $\frac{\sqrt{11}}{6} = \operatorname{sech}(x)$

$\Rightarrow \cosh(x) = \frac{6}{\sqrt{11}} = \frac{6\sqrt{11}}{11}$

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$$\sinh(1) = \frac{e^1 - e^{-1}}{2} = \frac{e - \frac{1}{e}}{2} = \frac{\frac{e^2-1}{e}}{2} = \frac{e^2-1}{2e}$$

Fine.

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

$$\sin(\sin^{-1}(x) + \cos^{-1}(x)) = \sin(\tfrac{\pi}{2})$$

$$= \sin(\sin^{-1}(x))\cos(\cos^{-1}(x)) + \sin(\cos^{-1}(x))\cos(\sin^{-1}(x)) = 1$$

$$= x \cdot x + \sqrt{1-x^2}\sqrt{1-x^2}$$

$$= x^2 + (1-x^2) = 1$$

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2} \pm 2\pi$$

Good scratch!

and dealing with $\sin(x)$, whose domain is and $\cos(x)$, which we restrict to $[0, \pi]$, so $\sin(\qquad)$ is assuming $(\qquad)$ is between 0 & $\frac{\pi}{2}$, inclusive. crappy.

Better than

$$\left.\begin{array}{l} \sin^{-1}(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \cos^{-1}(x) \in [0, \pi] \end{array}\right\} \sin^{-1}(x) + \cos^{-1}(x) \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}]$$

& only one value in there satisfies $\sin(\text{value}) = 1$, namely value $= \frac{\pi}{2}$.

$$\frac{d}{dx}\left[2^{3^{x^2}}\right]$$

$b=2$

$f(x)=3^{x^2}$

$$= \ln(2)\cdot 2^{3^{x^2}}\cdot \ln(3)\cdot 3^{x^2}\cdot 2x$$

$$\frac{d}{dx}\left[b^x\right]=\ln(b)\cdot b^x$$

$$\frac{d}{dx}\left[b^{f(x)}\right]=f'(x)\ln(b)\cdot b^x$$

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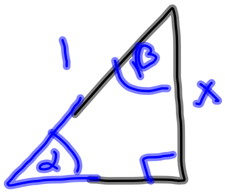
Proof by picture



$\cos\beta = \frac{x}{1}$

$\alpha = \sin^{-1}(x)$

$\beta = \cos^{-1}(x)$

$\alpha + \beta = \frac{\pi}{2}$, because they're complementary angles in a right triangle.

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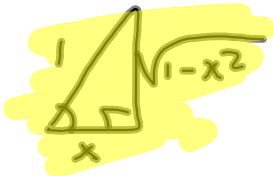
$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$



$$\sin\left(\sin^{-1}(x) + \cos^{-1}(x)\right) = \sin\left(\frac{\pi}{2}\right)$$

$$= \sin(\sin^{-1}(x))\cos(\cos^{-1}(x)) + \sin(\cos^{-1}(x))\cos(\sin^{-1}(x)) = 1$$

$$x \cdot x + \sqrt{1-x^2}\sqrt{1-x^2}$$

$$= x^2 + (1-x^2) = 1$$